Old & new co-ordinates

Aim lecture: A wise choice of co-ordinates can make life much easier. We give some examples showing how to make a linear change of co-ords to facilitate calculations.

Suppose we have old co-ords $(x_1, \ldots, x_n) \in \mathbb{F}^n$ & some new co-ords $(y_1, \ldots, y_n) \in \mathbb{F}^n$. They really lie in different copies of $\mathbb{F}^n$ & we need some notation to distinguish the two.

Notn

We let $\mathbb{F}^n_x$ be a copy of $\mathbb{F}^n$ whose co-ords are $(x_1, \ldots, x_n)$ & $\mathbb{F}^n_y$ be another copy of $\mathbb{F}^n$ whose co-ords are $(y_1, \ldots, y_n)$.
We let $GL_n(\mathbb{F})$ denote the set of invertible $n \times n$-matrices with entries in $\mathbb{F}$.

Change of co-ordinate matrix

Defn

A *change of co-ordinates matrix* (with say old co-ords $(x_1, \dots, x_n)$ & new co-ords $(y_1, \dots, y_n)$) is a co-ord system of the form $C : \mathbb{F}^n_y \longrightarrow \mathbb{F}^n_x$. Note the change of co-ordinates formula

$$(x_1, \dots, x_n)^T = C(y_1, \dots, y_n)^T, \quad (y_1, \dots, y_n)^T = C^{-1}(x_1, \dots, x_n)^T$$

Rem The possible change of co-ords matrices correspond to the elements of $GL_n(\mathbb{F})$.

E.g. Best to visualise an example geometrically. Suppose $C : \mathbb{R}^2_y \longrightarrow \mathbb{R}^2_x$ is given by the matrix

$$C = \begin{pmatrix} \frac{3}{5} & -\frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{pmatrix}$$

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Reflections via change of co-ordinate system

E.g. Use change of co-ordinates to find the matrix $A \in M_{22}(\mathbb{R})$ representing reflection about the line $x_2 = \frac{4}{3}x_1$.

A We use the co-ordinate system $C : \mathbb{R}^2_y \longrightarrow \mathbb{R}^2_x$ defined by the matrix

$$C = \begin{pmatrix} \frac{3}{5} & -\frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{pmatrix}.$$

In the new co-ords, the representing matrix is easy

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \mapsto \begin{pmatrix} y_1 \\ -y_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.$$

But our theory of matrix reprns also gives

$$C^{-1}AC = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Note

$$C^{-1}AC : \mathbb{R}^2_y \xrightarrow{C} \mathbb{R}^2_x \xrightarrow{A} \mathbb{R}^2_x \xrightarrow{C^{-1}} \mathbb{R}^2_y .$$

Hence $A =$

Linear maps & change of co-ords

The previous example illustrates the general principle below.

Prop

Let $\mathbb{F}^n_x$ be the space of old co-ords $(x_1, \ldots, x_n)$ & $\mathbb{F}^n_y$ be the space of new co-ords $(y_1, \ldots, y_n)$ which are related by the change of co-ords matrix $C : \mathbb{F}^n_y \longrightarrow \mathbb{F}^n_x$. If $A_x : \mathbb{F}^n_x \longrightarrow \mathbb{F}^n_x$ denotes a linear map in old co-ords & $A_y : \mathbb{F}^n_y \longrightarrow \mathbb{F}^n_y$ denotes the corresponding map in new co-ords then

$$A_y = C^{-1} A_x C, \quad A_x = C A_y C^{-1}$$

Rem It is useful to think of the composite

$$C A_y C^{-1} : \mathbb{F}^n_x \xrightarrow{C^{-1}} \mathbb{F}^n_y \xrightarrow{A_y} \mathbb{F}^n_y \xrightarrow{C} \mathbb{F}^n_x$$

as the sequence

1. 1st pass to new co-ords with C^{-1}
2. then use linear transformation A_y wrt new & easier co-ords $(y_1, \ldots, y_n)$.
3. finally use C to convert back to original co-ords.

Rotations via change of co-ords

E.g. Write down the matrix representing rotation about the axis $\mathbb{R}\,\mathbf{v}$ where $\mathbf{v} = \frac{1}{3}(1,2,2)^T$. Suppose the angle of rotation is θ & the direction is given by the right hand rule with thumb pointing in dirn $\mathbf{v}$.

A We first find a good co-ord system to use.

Rotation example cont'd

Note that the representing matrix in the new co-ords is

Swapping between two co-ordinate systems

Let $C_x : \mathbb{F}_x^n \longrightarrow V, C_y : \mathbb{F}_y^n \longrightarrow V$ be two co-ordinate systems. Then we can pass from x-co-ords to y-co-ords via $\mathbb{F}_x^n \xrightarrow{C_x} V \xrightarrow{C_y^{-1}} \mathbb{F}_y^n$. This suggests

Defn

The *change of co-ordinates matrix* is $C = C_y^{-1} \circ C_x$. Given the x-co-ords $\mathbf{x}$ of a vector $\mathbf{v} \in V$, its y-co-ords are $\mathbf{y} = C\mathbf{x}$.

E.g. Suppose we have new co-ord systems on $\mathbb{R}^2$ given by

$$C_x = \begin{pmatrix} 2 & 1 \\ 3 & 1 \end{pmatrix}, C_y = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}.$$

If the x-co-ords of $\mathbf{v} \in \mathbb{R}^2$ are $\binom{x_1}{x_2}$, what are the y-co-ords? What is the change of co-ords matrix?

Being careful with identifying vector spaces

Isomorphisms sometimes allow us to identify two vector spaces, but this can be confusing at times, & we should be more careful as follows. Suppose given isomorphisms $\Phi_V : V' \longrightarrow V, \Phi_W : W' \longrightarrow W$.

Defn

Given isomorphisms Φ_V, Φ_W above, the linear map $T : V \longrightarrow W$ induces a *corresponding linear map* $T' = \Phi_W^{-1} \circ T \circ \Phi_V : V' \longrightarrow W'$. If there is no confusion in treating Φ_V, Φ_W as identifications, then we may also make the identification $T' = T$.

E.g. Suppose $m = m_1 + \ldots + m_M, n = n_1 + \ldots + n_N$ so there are natural isomorphisms $\Phi_W : \mathbb{F}^m \longrightarrow \mathbb{F}^{m_1} \oplus \ldots \oplus \mathbb{F}^{m_M}$, $\Phi_V : \mathbb{F}^n \longrightarrow \mathbb{F}^{n_1} \oplus \ldots \oplus \mathbb{F}^{n_N}$ given by "removing" internal parentheses.

Given an $M \times N$-matrix of matrices defining a linear map $T : \mathbb{F}^{n_1} \oplus \ldots \oplus \mathbb{F}^{n_N} \longrightarrow \mathbb{F}^{m_1} \oplus \ldots \oplus \mathbb{F}^{m_M}$, the corresponding linear map $\Phi_W^{-1} \circ T \circ \Phi_V : \mathbb{F}^n \longrightarrow \mathbb{F}^m$ is just the big matrix obtained by "removing" internal parentheses.