Isomorphisms preserve linear concepts

Aim lecture: Given a vector space or linear map between vector space, the easiest way to compute with them is to use co-ordinate systems. We show how this works.

Prop 1

Let $T : V \longrightarrow W$ be an isomorphism of $\mathbb{F}$-spaces & $B \subseteq V$ be a finite subset.

1. For any subspace $V' \leq V$, we obtain an isomorphism by restriction $T|_{V'} : V' \longrightarrow T(V')$.
2. B is a spanning set for V iff $T(B)$ is a spanning set for W.
3. B is linearly independent iff $T(B)$ is.
4. B is a basis for V iff $T(B)$ is basis for W.

Rem An important case where we use this propn is where T is a co-ordinate system.

Proof. For 1) note that $T|_{V'}$ is surjective because we altered the co-domain to $T(V')$. It is injective since the eqn $T\mathbf{v} = \mathbf{w}$ still has unique solns (if at all). Finally, we know $T|_{V'}$ is linear so we are done.

Proof continued

Note that 4) follows from 2) & 3) put together.

2) ($\Longrightarrow$) Suppose $B = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ spans V. Hence for any $\mathbf{w} \in W$ we may write $T^{-1}\mathbf{w} = \sum_i \beta_i \mathbf{v}_i$ for some scalars $\beta_i \in \mathbb{F}$. Then $\mathbf{w} = \sum_i \beta_i T\mathbf{v}_i \in \text{Span}(T(B))$ so the forward implication holds. To prove the reverse implication, we just apply the ($\Longrightarrow$) result proved to $T^{-1} : W \longrightarrow V$ & the subset $T(B)$.

3) ($\Longrightarrow$) We prove the contrapositive & suppose $T(B)$ is linearly dependent so there is a non-trivial linear relation

$$\sum_i \beta_i T\mathbf{v}_i = \mathbf{0} \quad \text{for some } \beta_i \in \mathbb{F}$$

say with $\beta_j \neq 0$. Then B is also linearly dependent since applying T^{-1} to the above eqn gives the non-trivial linear relation $\sum_i \beta_i \mathbf{v}_i = \mathbf{0}$. As in 2), the converse is proved by applying the forward implication to T^{-1} & $T(B)$.

Example on finding a basis for a span

E.g. Use the co-ordinate system $C = (1 \quad x \quad x^2) : \mathbb{F}^3 \longrightarrow \mathbb{F}[x]_{\leq 2}$ to determine a basis and hence co-ord system for
$W = \text{Span}(1 + x + x^2, -1 + x - 2x^2, 2 + 4x + x^2)$. (You can do this question directly too without co-ordinates).
A Prop 2) shows that $C^{-1}(W) = \text{Span}$

From first year we know how to reduce this spanning set for $C^{-1}(W)$ to a basis.

Matrix representing a linear map wrt co-ordinate systems

Let $T : V \longrightarrow W$ be a linear map and $C_V : \mathbb{F}^n \longrightarrow V, C_W : \mathbb{F}^m \longrightarrow W$ be co-ordinate systems. Consider the composite map

$$C_W^{-1} \circ T \circ C_V : \mathbb{F}^n \xrightarrow{C_V} V \xrightarrow{T} W \xrightarrow{C_W^{-1}} \mathbb{F}^m .$$

This is a linear map from $\mathbb{F}^n \longrightarrow \mathbb{F}^m$ so can be represented by an $m \times n$-matrix in $M_{mn}(\mathbb{F})$.

Defn

The *matrix representing T wrt co-ordinate systems* C_V, C_W is the matrix giving the linear map $C_W^{-1} \circ T \circ C_V$.

Rem Recall the co-ordinate systems are given by row matrices $C_V = (\mathbf{v}_1 \dots \mathbf{v}_n) \in V^n, C_W = (\mathbf{w}_1 \dots \mathbf{w}_m) \in W^m$ which are essentially "ordered bases". In the literature, it is more common to speak of matrices representing T wrt ordered bases.

Rem Knowing the representing matrix $A = C_W^{-1} \circ T \circ C_V$ & the co-ordinate systems C_V, C_W allows you to recover all the information about T for $T = C_W \circ A \circ C_V^{-1}$. In particular, A will allow us to compute whatever we like about T.

Linear map induced by multiplication

Let X = set & consider the $\mathbb{F}$-space of functions $V = \text{Fun}(X, \mathbb{F})$.

Lemma

Fix a function $p(x) \in \text{Fun}(X, \mathbb{F})$. The map $p(x) : \text{Fun}(X, \mathbb{F}) \longrightarrow \text{Fun}(X, \mathbb{F}) : f(x) \mapsto p(x)f(x)$ is linear.

Proof. For $f(x), g(x) \in \text{Fun}(X, \mathbb{F}), \beta \in \mathbb{F}$, the distributive law

$$p(x)(f(x) + g(x)) = p(x)f(x) + p(x)g(x)$$

shows that the map $p(x)$ respects addn whilst the commutative & associative law

$$p(x)\beta f(x) = \beta p(x)f(x)$$

shows the map $p(x)$ respects scalar multn.

Rem The notation for this map is bad but common.

Example of finding representing matrix

E.g. Consider the linear map $T : \mathbb{R}[x]_{\leq 2} \longrightarrow \mathbb{R}[x]_{\leq 3}$ defined by $(Tf)(x) = (x-1)f'(x) - 2f(x)$ & the co-ordinate systems $C_1 = (1\ x\ x^2) : \mathbb{R}^3 \longrightarrow \mathbb{R}[x]_{\leq 2}, C_2 : (1\ x\ x^2\ x^3) : \mathbb{R}^4 \longrightarrow \mathbb{R}[x]_{\leq 3}$. Find the matrix representing T wrt C_1, C_2.

Rem Note that T is linear since $T = (x-1) \circ \frac{d}{dx} - 2\,\text{id}$ & we know $x - 1, \frac{d}{dx}, \text{id}$ are linear as are composites & linear combns.

A We consider $\mathbb{R}^3 \xrightarrow{C_1} \mathbb{R}[x]_{\leq 2} \xrightarrow{T} \mathbb{R}[x]_{\leq 3} \xrightarrow{C_2^{-1}} \mathbb{R}^4$.
We know from lecture 12, the representing matrix $A \in M_{43}(\mathbb{R})$ has i-th column $C_2^{-1} \circ T \circ C_1 \mathbf{e}_i$.
1st column is

Isomorphisms preserve kernels & images

The following allows us to compute kernels & images of linear maps from the representing matrix.

Prop 2

Let $T : V \longrightarrow W$ be a linear map & $C_1 : V' \longrightarrow V, C_2 : W' \longrightarrow W$ be isomorphisms. Let $T' = C_2^{-1} \circ T \circ C_1 : V' \longrightarrow W'$.

1. $\text{im } T = C_2(\text{im } T')$.
2. $\ker T = C_1(\ker T')$.

Proof. Note that $C_2 \circ T' = T \circ C_1$.
For 1), observe that since C_1 is onto we have $C_1(V') = V$. Thus

$$\text{im } T = T(V) = T(C_1(V')) = C_2(T'(V')) = C_2(\text{im } T').$$

For 2) we first show that $C_1(\ker T') \subseteq \ker T$ so let $\mathbf{v}' \in \ker T'$. Then $T(C_1\mathbf{v}') = C_2(T'\mathbf{v}') = C_1\mathbf{0} = \mathbf{0}$ so $C_1\mathbf{v}' \in \ker T$ & we must have $C_1(\ker T') \subseteq \ker T$. The reverse inclusion is proved similarly or by applying the inclusion already proved to C_1^{-1}, C_2^{-1} & T', T with roles reversed.

Example computing bases for kernels & images

Recall e.g. Consider the linear map $T : \mathbb{R}[x]_{\leq 2} \longrightarrow \mathbb{R}[x]_{\leq 3}$ defined by $(Tf)(x) = (x-1)f'(x) - 2f(x)$ & the co-ordinate systems $C_1 = (1\ x\ x^2) : \mathbb{R}^3 \longrightarrow \mathbb{R}[x]_{\leq 2}, C_2 : (1\ x\ x^2\ x^3) : \mathbb{R}^4 \longrightarrow \mathbb{R}[x]_{\leq 3}$. Compute $\ker T, \text{im } T$ by finding bases for them.

A Recall the representing matrix $C_2^{-1} \circ T \circ C_1$ is

$$A = \begin{pmatrix} -2 & -1 & 0 \\ 0 & -1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

A basis for $\ker A$ is

so by prop 1 & 2, a basis for $\ker T$ is
A basis for im A is

so by prop 1 & 2, a basis for im T is