Kernel

Aim lecture: We examine the kernel which measures the failure of uniqueness of solns to linear eqns. The concept of linear independence naturally arises. This in turn gives the concept of a basis which allows us to construct co-ordinate systems.

Recall For a fn $f : X \longrightarrow Y$ and subset $Y' \subseteq Y$, the *inverse image* of Y' is the set $f^{-1}(Y') = \{x \in X | f(x) \in Y'\}$.

Prop-Defn

Let $T : V \longrightarrow W$ be a linear map.

1. For any subspace $W' \subseteq W$, the inverse image $T^{-1}(W')$ is a subspace of V.
2. In particular, the *kernel* of T, defined to be $\ker T = T^{-1}(\mathbf{0})$ is a subspace of V.

Proof. Just check closure axioms noting $T\mathbf{0}_V = \mathbf{0}_W \Longrightarrow \mathbf{0} \in T^{-1}(W')$ and for $\mathbf{v}, \mathbf{v}' \in T^{-1}(W'), \beta \in F$, we know $\beta\mathbf{v} + \mathbf{v}' \in T^{-1}(W')$ since

$$T(\beta\mathbf{v} + \mathbf{v}') = \beta T\mathbf{v} + T\mathbf{v}' \in W'$$

by closure axioms for W'.

Examples

E.g. 1 We saw last lecture that the set of solns V to the DE $\frac{d^2y}{dx^2} + y = 0$ is a subspace of $C^\infty(\mathbb{R})$ by showing it is the span of some set of vectors. We can also see immediately it is a subspace by recognising it as the kernel of the linear map $\frac{d^2}{dx^2} + \mathrm{id} : C^\infty(\mathbb{R}) \longrightarrow C^\infty(\mathbb{R})$.

In fact, many subspaces naturally arise or can be described this way.
E.g. 2 Let $\mathbf{v} \in \mathbb{R}^n$. Then the hyperplane $\mathbf{v}^\perp$ is a subspace of $\mathbb{R}^n$ since it is the kernel of the following map,

Kernel measures failure of injectivity

You should already know the following.

Prop

Let $T : V \longrightarrow W$ be a linear map & $\mathbf{w} \in W$.

1. Given a particular soln $\mathbf{v} = \mathbf{v}_p$ to the eqn $T\mathbf{v} = \mathbf{w}$, we obtain the complete set of solns as $T^{-1}(\mathbf{w}) = \{\mathbf{v}_p + \mathbf{v}_h | \mathbf{v}_h \in \ker T\}$.
2. In particular, the soln to $T\mathbf{v} = \mathbf{w}$ is unique (assuming it exists) iff $\ker T = \mathbf{0}$.
3. T is injective iff $\ker T = \mathbf{0}$.

Proof. 2)& 3) immediately follow from 1) which we now prove.
Given $\mathbf{v}_h \in \ker T$ note that

$$T(\mathbf{v}_p + \mathbf{v}_h) = T\mathbf{v}_p + T\mathbf{v}_h = \mathbf{w} + \mathbf{0} = \mathbf{w}$$

so $\mathbf{v}_p + \mathbf{v}_h \in T^{-1}(\mathbf{w})$. Hence it remains to show that any soln $\mathbf{v} \in T^{-1}(\mathbf{w})$ has the form $\mathbf{v}_p + \mathbf{v}_h$ for some $\mathbf{v}_h \in \ker T$.
Now $\mathbf{v}_h = \mathbf{v} - \mathbf{v}_p \in \ker T$ since

$$T\mathbf{v}_h = T(\mathbf{v} - \mathbf{v}_p) = T\mathbf{v} - T\mathbf{v}_p = \mathbf{w} - \mathbf{w} = \mathbf{0}.$$

Thus $\mathbf{v} = \mathbf{v}_p + \mathbf{v}_h$ has the desired form.

Injectivity of linear maps $C : \mathbb{F}^n \longrightarrow V$

We have seen that any linear map $C : \mathbb{F}^n \longrightarrow V$ determined by the row matrix $(\mathbf{v}_1 \dots \mathbf{v}_n) \in V^n$ is surjective iff $V = \text{Span}(\mathbf{v}_1, \dots, \mathbf{v}_n)$. We determine the condition for injectivity below.

Prop-Defn

With the above notn, the map C is injective iff for any $\beta_1, \dots, \beta_n \in \mathbb{F}$,

$$\beta_1 \mathbf{v}_1 + \dots + \beta_n \mathbf{v}_n = \mathbf{0} \Longrightarrow 0 = \beta_1 = \dots = \beta_n.$$

In other words, the only linear combn of the $\mathbf{v}_i$'s which is zero, is the trivial linear combn. In this case, we say that the set $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is *linearly independent*. Otherwise, we say it is *linearly dependent*.

Proof. Clear.

Ordered bases & co-ordinate systems

We now give a way of constructing co-ordinate systems.

Defn

Let $V = \mathbb{F}$-space. A *basis* for V is a linearly independent spanning set for V.

From the results we have already proved, we immediately see

Prop-Defn

Let $C : \mathbb{F}^n \longrightarrow V$ be the linear map given by the row matrix $(\mathbf{v}_1 \dots \mathbf{v}_n) \in V^n$. Then C defined a co-ordinate system iff $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is a basis for V. In other words, co-ordinate systems correspond to *ordered bases* for V. The *co-ordinates* of $\mathbf{v} \in V$ wrt C is $C^{-1}\mathbf{v} = (x_1, \dots, x_n)^T$. In this case,

$$\mathbf{v} = x_1\mathbf{v}_1 + \dots + x_n\mathbf{v}_n$$

is the unique way of writing $\mathbf{v}$ as a linear combn of $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$.

Note that for a co-ord system C as above, the corresp basis is its sets of entries which are given by $C\mathbf{e_j}, j = 1, \dots n$ i.e. it's the image of the standard basis on $\mathbb{F}^n$.

Example of co-ordinate systems

E.g. Show that the matrix $C = (1+x \quad 2+3x) : \mathbb{C}^2 \longrightarrow \mathbb{C}[x]_{\leq 1}$ defines a co-ordinate system on $\mathbb{C}[x]_{\leq 1}$. Find the co-ordinates of $3+4x$ wrt C.

Computing bases/ co-ord systems for matrices

Recall from MATH1241/1251 how to find bases for kernels of matrices.
E.g. Find a basis and hence co-ord system for the kernel of the matrix

$$A = \begin{pmatrix} 1 & 1 & -1 & 0 \\ 2 & 1 & 0 & 1 \\ 3 & 2 & -1 & 1 \end{pmatrix}.$$

Co-ordinate systems of direct sums.

Prop

1. For $i = 1, \ldots, r$, let $T_i : V_i \longrightarrow W_i$ be isomorphisms. Then $\oplus_i T_i : \oplus_i V_i \longrightarrow \oplus_i W_i$ is also an isomorphism.
2. In particular, given co-ordinate systems $C_i : \mathbb{F}^{n_i} \longrightarrow W_i$, we obtain a co-ordinate system $\oplus_i C_i : \mathbb{F}^n \longrightarrow \oplus_i W_i$ where $n = \sum_i n_i$.
3. Consider an internal direct sum $W = \oplus_i W_i$ & bases $B_i \subset W_i$ for W_i. Then $\cup_i B_i$ is a basis for W.

Proof. 1) We need only show $\oplus_i T_i^{-1}$ is the inverse for $\oplus_i T_i$. For ease of writing we do this when $r = 2$.

$$(T_1 \oplus T_2) \circ (T_1^{-1} \oplus T_2^{-1}) = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix} \begin{pmatrix} T_1^{-1} & 0 \\ 0 & T_2^{-1} \end{pmatrix} =$$

Sim, $(T_1^{-1} \oplus T_2^{-1}) \circ (T_1 \oplus T_2) = \text{id}$ so 1) holds.

It's clear 1) $\Longrightarrow$ 2). Also 3) follows from 2) as the basis corresp to $(\oplus C_i)$ consist of the $(\oplus C_i)\mathbf{e}_j, j = 1, \ldots, n$.

A matrix example

E.g. We have seen already that $V = M_{22}(\mathbb{R})$ is the internal direct sum of the subspaces V^+, V^- of symmetric & anti-symmteric matrices resp.

Now any symmetric (resp anti-symmetric) matrix can be written uniquely in the form

$$\begin{pmatrix} a & b \\ b & c \end{pmatrix}, \quad \text{resp} \begin{pmatrix} 0 & d \\ -d & 0 \end{pmatrix}$$

so we obtain the following bases for V^+, V^-, V.

Alternate characterisation of linear dependence

Prop

A finite set $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ of vectors is linearly dependent iff we can write one of them as a linear combn of the others.

Proof. Suppose that S is lin depedent so there is a non-trivial linear relation say $\beta_1 \mathbf{v}_1 + \ldots \beta_n \mathbf{v}_n = \mathbf{0}$ with say $\beta_i \neq 0$. Then $\mathbf{v}_i$ is a linear combn of the others as we may re-write

$$\mathbf{v}_i = -\sum_{j \neq i} \beta_i^{-1} \beta_j \mathbf{v}_j.$$

Reversing the above computation gives the converse.

Alternate characterisation of bases

Theorem

Let $V = \mathbb{F}$-space & $B \subseteq V$ be finite. The following are equiv conds on B.

1. *B is a basis for V.*
2. *B is a minimal spanning set for V.*
3. *B is a maximal linearly independent set in the sense that, B is linearly independent but B' is linearly dependent for any set B' strictly containing B.*

Proof. The equivalence of 1) & 2) follows from the alternate characterisation of linear dependence. The proof of 1) $\iff$ 3) follows easily (ex) from

Lemma

Suppose $B = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is lin independent & let $\mathbf{v} \in V$. Then $B \cup \{\mathbf{v}\}$ is lin indep iff $\mathbf{v} \notin Span(B)$.

proof continued

To see ($\Longrightarrow$), suppose $B \cup \{\mathbf{v}\}$ is lin indep but $\mathbf{v} \in \text{Span}(B)$. Then $B \cup \{\mathbf{v}\}$ must be linearly dependent by our alternate charn of linear dependence and we obtain a contradiction. Hence $\mathbf{v} \notin \text{Span}(B)$.
For ($\Longleftarrow$), suppose that $\mathbf{v} \notin \text{Span}(B)$ but $B \cup \{\mathbf{v}\}$ is lin dependent. Hence there is a non-trivial linear relation of form

$$\beta\mathbf{v} + \beta_1\mathbf{v}_1 + \ldots + \beta_n\mathbf{v}_n = \mathbf{0}.$$

Now $\beta \neq 0$ as B is linearly independent. Hence we may solve to see $\mathbf{v}$ is a linear combn of the $\mathbf{v}_i$ so lies in $\text{Span}(B)$. Again we obtain a contradiction so $B \cup \{\mathbf{v}\}$ is lin indep.

Corollary

Any vector space that is spanned by a finite set S has a basis consisting of a subset of S.