The image of a linear map

Aim lecture: We relate the important notions of the span of a set of vectors and the image of a linear map. They can be used to give "parametric forms" for vector spaces & in particular, help construct co-ordinate systems.

Prop-Defn

Let $T : V \longrightarrow W$ be a linear map.

1. For any $V' \leq V$ we have $T(V') = \{T\mathbf{v}' \in W | \mathbf{v}' \in V'\}$ is a subspace of W.
2. In particular, the *image of* T, defined to be im $T = T(V)$ is a subspace of W.

Proof. We just need to check closure axioms. Note $\mathbf{0}_W = T\mathbf{0}_V \in T(V')$ so $T(V')$ is non-empty. Also, for $\mathbf{v}', \mathbf{v}'' \in V', \beta \in \mathbb{F}$ we have

$$\beta T\mathbf{v}' + T\mathbf{v}'' = T(\beta\mathbf{v}' + \mathbf{v}'') \in T(V')$$

so closure axioms hold & propn-defn is proved.

1. Note that surjectivity of T just means that im $T = W$.
2. $T(V')$ is an example of a set defined by "parametric form" where the parameter is $\mathbf{v}' \in V'$.

The image of a linear map $T : \mathbb{F}^n \longrightarrow V$

Prop-Defn

Let $C : \mathbb{F}^n \longrightarrow V$ be a linear map given by the row matrix $(\mathbf{v}_1 \dots \mathbf{v}_n) \in V^n$. Then

$$\text{im } T = \mathbb{F}\,\mathbf{v}_1 + \dots + \mathbb{F}\,\mathbf{v}_n.$$

We define the $\mathbb{F}$*-span of* $\mathbf{v}_1, \dots, \mathbf{v}_n$ to be the subspace im T of V and denote it $\text{Span}(\mathbf{v}_1, \dots, \mathbf{v}_n)$. In other words, the span is the set of all linear combns of $\mathbf{v}_1, \dots, \mathbf{v}_n$.

Proof. Just calculate

Note that the span does not depend on the order of the vectors so it makes sense to make the following

Defn

We say that $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is a *spanning set* for V or *spans* V if $\text{Span}(\mathbf{v}_1, \dots, \mathbf{v}_n) = V$ or equiv, the map $C : \mathbb{F}^n \longrightarrow V$ above is surjective. In this case we say V is *finitely spanned*.

First year example

Make sure you remember from MATH1241/1251, how to do the following question.

E.g. Does $S = \{1 + 2x^2, 1 - 2x, 2 + x + 5x^2\}$ span $\mathbb{R}[x]_{\leq 2}$?

A Since $S \subset \mathbb{R}[x]_{\leq 2}$, need only check given any $p(x) = a + bx + cx^2 \in \mathbb{R}[x]_{\leq 2}$, is it true that $p(x) \in \text{Span}(S)$, i.e. can we always solve

$$\alpha(1 + 2x^2) + \beta(1 - 2x) + \gamma(2 + x + 5x^2) = a + bx + cx^2.$$

Another Example

E.g. Let V be the set of solns (in $C^\infty(\mathbb{R})$) to the DE $\frac{d^2y}{dx^2} + y = 0$. Show that V is a subspace of $C^\infty(\mathbb{R})$ by showing it is the span of some set of vectors. Also, find a co-ordinate system for V.

Span of infinite sets

We can generalise the old defn of span to (possibly) infinite sets.

Defn

Let V be a vector space and $S \subseteq V$. The *span* of S, denoted $\text{Span}(S)$, is defined to be the set of all linear combns of elements of S, i.e. set of all vectors of the form $\beta_1\mathbf{v}_1 + \ldots + \beta_n\mathbf{v}_n$ where $\mathbf{v}_1, \ldots, \mathbf{v}_n \in S$.

Prop

With above notn, $\text{Span}(S)$ is the unique smallest subspace of V containing S. More precisely, any subspace $W \leq V$ which contains S must also contain $\text{Span}(S)$.

Proof. One can check closure axioms to see $\text{Span}(S)$ in this more general setting is still a subspace. It clearly contains S. Suppose $W \leq V$ contains S. It closed under linear combns so in particular, contains all linear combns of elts of S, i.e. it contains $\text{Span}(S)$.

Span as an "increasing" function of S

We look at the question of how $\text{Span}(S)$ changes as you vary S.

Prop

Let $V = \mathbb{F}$-space and $S \subseteq V, \mathbf{v} \in V$.

1. $\text{Span}(S) \subseteq \text{Span}(S \cup \{\mathbf{v}\})$.
2. Equality in 1) holds iff $\mathbf{v} \in \text{Span}(S)$.

Proof. Note that $\text{Span}(S \cup \{\mathbf{v}\})$ is a subspace containing S so the minimality property of $\text{Span}(S)$ ensures 1) holds.
Suppose now equality holds in 1). Then

$$\mathbf{v} \in \text{Span}(S \cup \{\mathbf{v}\}) = \text{Span}(S).$$

Conversely suppose that $\mathbf{v} \in \text{Span}(S)$. Then $\text{Span}(S)$ is a subspace containing $S \cup \{\mathbf{v}\}$ so must contain $\text{Span}(S \cup \{\mathbf{v}\})$. Hence equality holds in 1).

Span in $\mathbb{R}^3$

Let start with a non-zero $\mathbf{v}_1 \in \mathbb{R}^3$ and let $S = \{\mathbf{v}_1\}$. Then $\text{Span}(S) = \mathbb{R}\,\mathbf{v}_1$ is a line.

Now let's add $\mathbf{v}_2 \in \mathbb{R}^3$ to S so $S = \{\mathbf{v}_1, \mathbf{v}_2\}$. There are 2 cases.

Minimal spanning sets

Defn

We say that a spanning set S for an $\mathbb{F}$-space V is *minimal* if any proper subset $S' \subset S$ does not span V. Equivalently, by the previous propn, S is a minimal spanning set if each $\mathbf{v} \in S$ is not contained in $\text{Span}(S - \{\mathbf{v}\})$ i.e. $\mathbf{v}$ is not a linear combn of other vectors in S.

Any finitely spanned vector space has a minimal spanning set. Indeed, start with any finite spanning set S. Then either it is minimal or we can find some $\mathbf{v} \in S$ which is a linear combn of other vectors in S. We throw away $\mathbf{v}$ and repeat until a minimal one is found. This algorithm terminates because S is finite.

All vector spaces have minimal spanning sets, but the proof in general requires Zorn's lemma so will be omitted.

Example on finding minimal spanning sets

E.g. Find a minimal spanning set for Span(S) where
$S = \{p_1(x) = 3 + x, p_2(x) = x + x^2, p_3(x) = 3 + 2x + x^2\}$.