Matrix of linear maps. Matrix-vector product

Aim lecture: Introduce matrices of linear maps as a way of understanding more complicated linear maps.

Consider direct sums of $\mathbb{F}$-spaces $V = \oplus_{j=1}^m V_j$, $W = \oplus_{i=1}^n W_i$.

Notn

Write $(L(V_j, W_i))_{ij}$ for the direct sum $\oplus_{j=1}^m \oplus_{i=1}^n L(V_j, W_i)$ except the elements are not written as column vectors but in an $n \times m$-matrix so has form $T = (T_{ij})_{ij}$ where the entries $T_{ij} : V_j \longrightarrow W_i$ are linear & $i = 1, \ldots, n, j = 1, \ldots, m$.

Given $\mathbf{v} = (\mathbf{v}_1, \ldots, \mathbf{v}_m)^T \in V$ and $T = (T_{ij})_{ij} \in (L(V_j, W_i))_{ij}$ we can define the *matrix-vector product* $T\mathbf{v}$ which is the element $\mathbf{w}$ of W whose i-th entry is

$$\mathbf{w}_i = \sum_{j=1}^m T_{ij}\mathbf{v}_j \in W_i.$$

Evaluation map & examples of matrix-vector product

It is an easy exercise to prove

Prop-Defn

For a set X & $x \in X$, the *evaluation at x map* $\mathrm{ev}_x : \mathrm{Fun}(X, \mathbb{F}) \longrightarrow \mathbb{F}$ defined by $\mathrm{ev}_x(f) = f(x)$ is linear.

E.g.1 We can put the lin maps $\frac{d}{dx} : \mathbb{R}[x] \longrightarrow \mathbb{R}[x], \mathrm{ev}_0 : \mathbb{R}[x] \longrightarrow \mathbb{R}, 2\,\mathrm{id} : \mathbb{R} \longrightarrow \mathbb{R}$ into a matrix to define

$$\begin{pmatrix} \frac{d}{dx} & 0 \\ \mathrm{ev}_0 & 2\,\mathrm{id} \end{pmatrix} \begin{pmatrix} f(x) \\ \beta \end{pmatrix} =$$

for $f(x) \in \mathbb{R}[x], \beta \in \mathbb{F}$. Note the 2×2-matrix above defines a map $\mathbb{R}[x] \oplus \mathbb{R} \longrightarrow \mathbb{R}[x] \oplus \mathbb{R}$.

E.g.2 We may re-write the initial value problem

$$\frac{d^2y}{dy^2} - 4y = \sin x, \quad y(0) = 3$$

as the single matrix eqn

Prop

Let $V = \mathbb{F}$-space.

1. For any $\mathbf{v} \in V$, the map $T_{\mathbf{v}} : \mathbb{F} \longrightarrow V$ defined by $T_{\mathbf{v}}\beta = \beta\mathbf{v}$ is linear.
2. Every $\mathbb{F}$-lin map $T : \mathbb{F} \longrightarrow V$ can be written uniquely in the form $T_{\mathbf{v}}$ for some $\mathbf{v} \in V$ as above.
3. In fancier language, there is a natural isomorphism $V \longrightarrow L(\mathbb{F}, V) : \mathbf{v} \mapsto T_{\mathbf{v}}$.

Proof. For 1) just note for $\gamma, \beta, \beta' \in \mathbb{F}$ we have
$T_{\mathbf{v}}(\gamma\beta + \beta') = (\gamma\beta + \beta')\mathbf{v} = \gamma\beta\mathbf{v} + \beta'\mathbf{v} = \gamma T_{\mathbf{v}}\beta + T_{\mathbf{v}}\beta'$.
For 2), given lin $T : \mathbb{F} \longrightarrow V$ we let $\mathbf{v} = T(1)$ and note $T = T_{\mathbf{v}}$ since for any $\gamma \in \mathbb{F}$ we have

$$T(\gamma) = \gamma T(1) = \gamma\mathbf{v} = T_{\mathbf{v}}\gamma.$$

No other choice of vector $\mathbf{v}$ works so it is unique. You can check as ex that the bijection $V \longrightarrow L(\mathbb{F}, V) : \mathbf{v} \mapsto T_{\mathbf{v}}$ is indeed linear so 3) follows.

Upshot: Hence we will often identify $L(\mathbb{F}, V)$ with V & in particular, $L(\mathbb{F}, \mathbb{F})$ with $\mathbb{F}$

Connection with old matrix product

Consider now the situation $V = \mathbb{F}^n = \oplus_{j=1}^n \mathbb{F}, W = \mathbb{F}^m = \oplus_{i=1}^m \mathbb{F}$. Then a matrix $A \in M_{mn}(\mathbb{F})$ in the old sense is also a matrix in $(L(\mathbb{F}, \mathbb{F}))_{ij}$ in the new sense. Moreover, for $\mathbf{v} \in V$ the two defns (old & new) of $A\mathbf{v}$ define the same element of W.

Linearity of matrix-vector product

Consider direct sums of $\mathbb{F}$-spaces $V = \oplus_{j=1}^{m} V_j, W = \oplus_{i=1}^{n} W_i$. Below we write $(\mathbf{v}_j)_j$ as an abbreviated form for $(\mathbf{v}_1, \ldots, \mathbf{v}_m)^T$.

Prop

Consider a matrix of linear maps $T = (T_{ij})_{ij} \in (L(V_j, W_i))_{ij}$. Then (abusing notn) the associated map $T : V \longrightarrow W : \mathbf{v} \mapsto T\mathbf{v}$ is linear.

Proof. For $\mathbf{v} = (\mathbf{v}_j)_j, \mathbf{v}' = (\mathbf{v}'_j)_j \in V, \beta \in \mathbb{F}$. Then

$$\begin{aligned} T(\beta\mathbf{v} + \mathbf{v}') &= T(\beta\mathbf{v}_j + \mathbf{v}'_j)_j = (\sum_j T_{ij}(\beta\mathbf{v}_j + \mathbf{v}'_j))_i \\ &= (\sum_j(\beta T_{ij}\mathbf{v}_j + T_{ij}\mathbf{v}'_j))_i = (\beta\sum_j T_{ij}\mathbf{v}_j + \sum_j T_{ij}\mathbf{v}'_j)_i \\ &= \beta T\mathbf{v} + T\mathbf{v}'. \end{aligned}$$

This completes the proof.

Special case of row matrices and linear maps $\mathbb{F}^n \longrightarrow V$

Let $V = \mathbb{F}$-space. Identifying $V = L(\mathbb{F}, V)$ we get

Cor

Consider a row matrix $A = (\mathbf{v}_1\ \mathbf{v}_2 \ldots \mathbf{v}_n) \in V^n$. Then A defines a linear map $T : \mathbb{F}^n \longrightarrow V$ by

$$T(\beta_1, \ldots, \beta_n)^T = A(\beta_1, \ldots, \beta_n)^T = \beta_1\mathbf{v}_1 + \ldots + \beta_n\mathbf{v}_n.$$

E.g. Show that the row matrix $C = (1 + 2x \quad x) : \mathbb{R}^2 \longrightarrow \mathbb{R}[x]_{\leq 1}$ defines a co-ord system on $\mathbb{R}[x]_{\leq 1}$.

A Cor $\Longrightarrow$ C defines a linear map so it suffices to check C is bijective i.e. for any $a, b \in \mathbb{R}$, there is a unique soln to

$$a + bx = C\begin{pmatrix}\alpha \\ \beta\end{pmatrix} = \alpha(1 + 2x) + \beta x$$

Identifying matrices with associated linear maps

Rem We will often confuse a matrix as above with the associated linear map induced by the matrix product. What permits us to do this is the following fact.

Fact

Distinct matrices induce distinct linear maps.

Proof. Follows by generalising the example below & will be proved in full generality in proof of matrix reprn thm.

E.g. Consider the map

$$T : \mathbb{F}[x] \oplus \mathbb{F} \longrightarrow \mathbb{F} \oplus \mathbb{F} : \begin{pmatrix} f(x) \\ \beta \end{pmatrix} \mapsto \begin{pmatrix} f(1) + 2\beta \\ f(0) - 3\beta \end{pmatrix}.$$

1. Show that T is linear by finding a 2×2-matrix of linear maps inducing it.
2. Explain why the 2×2-matrix found above is uniquely determined by T.

Direct sums of linear maps

Prop-Defn

For $i = 1, \dots, m$, let $T_i : V_i \longrightarrow W_i$ be linear maps. We define the map
$\oplus_{i=1}^m T_i = T_1 \oplus \dots \oplus T_m : V_1 \oplus \dots \oplus V_m \longrightarrow W_1 \oplus \dots \oplus W_m$
by $(\oplus_{i=1}^m T_i)(\mathbf{v}_1, \dots, \mathbf{v}_m)^T = (T_1\mathbf{v}_1, \dots, T_m\mathbf{v}_m)^T$. It is a linear map called the *direct sum* of the T_i.

Proof. Linearity follows from the fact that $\oplus_{i=1}^m T_i$ corresponds to the "diagonal matrix" $(T_{ij})_{ij} \in (L(V_j, W_i))_{ij}$ with $T_{ii} = T_i$ & $T_{ij} = 0$ if $i \neq j$. This is easily seen from any

E.g. $T_1 = \frac{d}{dx} : C^\infty(\mathbb{R}) \longrightarrow C^\infty(\mathbb{R}), T_2 = (2 \quad 3) : \mathbb{R}^2 \longrightarrow \mathbb{R}.$

Example of matrices of matrices

Matrices define linear maps so can be the entries of a matrix of linear maps too.
E.g. Consider the 4 matrices

$$A_{11} = \begin{pmatrix} 1 & 2 \\ 5 & 6 \end{pmatrix}, A_{12} = \begin{pmatrix} 3 & 4 \\ 7 & 8 \end{pmatrix}, A_{21} = \begin{pmatrix} 9 & 10 \end{pmatrix}, A_{22} = \begin{pmatrix} 11 & 12 \end{pmatrix}$$

Then identifying $\mathbb{F}^4 = \mathbb{F}^2 \oplus \mathbb{F}^2, \mathbb{F}^3 = \mathbb{F}^2 \oplus \mathbb{F}^1$ we find

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

which is the linear map associated to the 3×4-matrix obtained by "removing internal brackets".

General result concerning matrices of matrices

We can generalise this result as follows. Let $m = m_1 + \ldots + m_M, n = n_1 + \ldots + n_N$. We identify $\mathbb{F}^m = \mathbb{F}^{m_1} \oplus \ldots \oplus \mathbb{F}^{m_M}, \mathbb{F}^n = \mathbb{F}^{n_1} \oplus \ldots \oplus \mathbb{F}^{n_N}$.

Prop

For $i = 1, \ldots, M, j = 1, \ldots, N$ let $A_{ij} \in M_{m_i n_j}$ so that we may form the $M \times N$-matrix $(A_{ij})_{ij}$ of matrices. The associated linear map from $\mathbb{F}^n \longrightarrow \mathbb{F}^m$ is induced by the $m \times n$-matrix obtained from $(A_{ij})_{ij}$ by "removing internal brackets".

Proof omitted. The proof is notationally messy but easy and follows as in the previous example.

Motto A matrix of matrices is really just one big fat matrix (of scalars)!