Direct sums

Aim lecture: Examine a method of constructing new vector spaces from old & conversely, decomposing vector spaces into simpler ones.

Let $V_1, \ldots, V_r$ be $\mathbb{F}$-spaces and consider the set of r-tuples $\bigoplus_{i=1}^{r} V_i = V_1 \oplus V_2 \oplus \ldots \oplus V_r = \{(\mathbf{v}_1, \ldots, \mathbf{v}_r)^T | \mathbf{v}_i \in V_i, \text{for all } i\}$.

Prop-Defn

$V_1 \oplus \ldots \oplus V_r$ is an $\mathbb{F}$-space when endowed with co-ordinatewise addn & scalar multn i.e.

1. $(\mathbf{v}_1, \ldots, \mathbf{v}_r)^T + (\mathbf{v}'_1, \ldots, \mathbf{v}'_r)^T = (\mathbf{v}_1 + \mathbf{v}'_1, \ldots, \mathbf{v}_r + \mathbf{v}'_r)^T$.
2. For $\beta \in \mathbb{F}$, $\beta(\mathbf{v}_1, \ldots, \mathbf{v}_r)^T = (\beta\mathbf{v}_1, \ldots, \beta\mathbf{v}_r)^T$.

It is called the *(external) direct sum* of $V_1, \ldots, V_r$.

Proof. Sim check of axioms as $\mathbb{F}^n$.
Notn For an $\mathbb{F}$-space V we write $V^n = \oplus_{i=1}^{n} V$ for the direct sum of n copies of V. This agrees with old notn $\mathbb{F}^n$ if $\mathbb{F}$ denotes $\mathbb{F}^1$.

Example

We work in the $\mathbb{C}$-space $V = \mathbb{C}[x] \oplus \mathbb{C}$. Show that $(2 - x, 1) \in V$ lies in the subspace $W = \mathbb{C}(1 + 2x, 3) + \mathbb{C}(3 + x, 4) < V$.

Associativity

E.g. For $m, n \in \mathbb{N}$, there is an obvious isomorphism $T : \mathbb{F}^m \oplus \mathbb{F}^n \longrightarrow \mathbb{F}^{m+n}$ defined by
$T((\beta_1, \dots, \beta_m)^T, (\gamma_1, \dots, \gamma_n)^T)^T = (\beta_1, \dots, \beta_m, \gamma_1, \dots, \gamma_n)^T$.
Note that there are infinitely many isomorphisms from $\mathbb{F}^m \oplus \mathbb{F}^n \longrightarrow \mathbb{F}^{m+n}$ but this is the only "natural" one so we will use it to identify the two $\mathbb{F}$-spaces & write, abusing notn, $\mathbb{F}^m \oplus \mathbb{F}^n = \mathbb{F}^{m+n}$. Care of course must be taken.
Generalising the idea in this example

Prop

For $\mathbb{F}$-spaces V, W, X we have natural isomorphisms

$$(V \oplus W) \oplus X \simeq V \oplus W \oplus X \simeq V \oplus (W \oplus X).$$

Henceforth, we will identify the three $\mathbb{F}$-spaces above with the usual caveat.

Simple properties of direct sums

Let $V_1, \ldots, V_r$ be $\mathbb{F}$-spaces with zeros $\mathbf{0}_{V_1}, \ldots \mathbf{0}_{V_r}$.

Prop

Let $V = V_1 \oplus \ldots \oplus V_r$.

1. The zero of V is $\mathbf{0}_V = (\mathbf{0}_{V_1}, \ldots, \mathbf{0}_{V_r})^T$.
2. $V(i) = \mathbf{0} \oplus \ldots \mathbf{0} \oplus V_i \oplus \mathbf{0} \oplus \ldots \oplus \mathbf{0}$ is a subspace of V isomorphic to V_i called the *canonical copy of V_i in V*.
3. $V = \sum_{i=1}^{r} V(i)$

Rem 2) & 3) mean V is a sum of subspaces isomorphic to the V_i.
Proof. For 2), note that $T : V_i \longrightarrow V(i) : \mathbf{v}_i \mapsto (\mathbf{0}, \ldots \mathbf{0}, \mathbf{v}_i, \mathbf{0}, \ldots, \mathbf{0})$ is the obvious isomorphism. 3) is clear.
For 1)

Internal direct sums

Previous propn suggests a strong relationship between sums & direct sums.

Prop-Defn

Let $V = \mathbb{F}$-space & $V', V'' \leq V$. We say that the sum $V' + V''$ is *direct* if $V' \cap V'' = \mathbf{0}$. In this case, we have a natural isomorphism

$$T : V' \oplus V'' \longrightarrow V' + V'' : \begin{pmatrix} \mathbf{v}' \\ \mathbf{v}'' \end{pmatrix} \mapsto \mathbf{v}' + \mathbf{v}''.$$

Abusing notn, we write with the usual caveat $V' + V'' = V' \oplus V''$. In particular, in this case, every element of $V' + V''$ can be written uniquely in the form $\mathbf{v}' + \mathbf{v}''$ with $\mathbf{v}' \in V', \mathbf{v}'' \in V''$.

Proof. We check linearity first. Consider $\binom{\mathbf{v}'_1}{\mathbf{v}''_1}, \binom{\mathbf{v}'_2}{\mathbf{v}''_2} \in V' \oplus V''$ & $\beta \in \mathbb{F}$.

Proof cont'd

By defn of sum, T is surjective (i.e. onto) so we need only check it is injective (1-1). Suppose that $T\binom{\mathbf{v}_1'}{\mathbf{v}_1''} = T\binom{\mathbf{v}_2'}{\mathbf{v}_2''}$.
Then $\mathbf{v}_1' + \mathbf{v}_1'' = \mathbf{v}_2' + \mathbf{v}_2''$ so

$$\mathbf{v}_1' - \mathbf{v}_2' = \mathbf{v}_2'' - \mathbf{v}_1'' \in V' \cap V'' = \mathbf{0}.$$

Hence $\mathbf{v}_1' = \mathbf{v}_2' \quad \& \quad \mathbf{v}_1'' = \mathbf{v}_2''$ so $\binom{\mathbf{v}_1'}{\mathbf{v}_1''} = \binom{\mathbf{v}_2'}{\mathbf{v}_2''}$ and T is 1-1.
This completes the proof.

Defn

If $V', V'' \leq V$ with $V = V' \oplus V''$, then we will say V'' is a (vector space) *complement* to V' in V.

A geometric example

Let $\mathbf{v} \in \mathbb{R}^3$ be non-zero. Consider the plane $V' = \mathbf{v}^{\perp}$ thru $\mathbf{0}$ perpendicular to $\mathbf{v}$ & line $V'' = \mathbb{R}\,\mathbf{v}$. Then $\mathbb{R}^3 = V' \oplus V''$ so V'' is a complement of V' & vice versa. Condns easy to see pictorially.

An example

Let $V = \mathbb{R}^3, V' = \mathbb{R}(2,0,0)^T + \mathbb{R}(3,4,0)^T, V'' = \mathbb{R}(1,1,-1)^T$. Show $V = V' \oplus V''$.

A matrix example

Defn

Let $A \in M_{nn}(\mathbb{F})$. We say that A is *symmetric* if $A^T = A$ and *anti-symmetric* if $A^T = -A$.

Let $V = M_{nn}(\mathbb{R})$ & V^+, V^- be the subspaces consisting of symmetric & anti-symmetric matrices resp.
Now $V^+ \cap V^- = \mathbf{0}$ since if A is both symm & anti-symm then

$$A^T = A = -A^T \Longrightarrow 2A^T = \mathbf{0} \Longrightarrow A = \mathbf{0}.$$

Moreover, $V^+ + V^- = V$ since we can always write

$$A =$$

Hence $M_{nn}(\mathbb{R}) = V^+ \oplus V^-$ & every real square matrix can be written uniquely as a sum of a symmetric matrix & an anti-symmetric one.

Don't confuse the internal and external direct sum

- You can take the external direct sum of any two $\mathbb{F}$-spaces, but the internal direct sum only applies to subspaces of a given vector space.
- The internal direct sum is a special type of sum.
- If you have two subspaces, you can construct both the external direct sum and the sum. If the sum happens to be direct, then it is said to be the internal direct sum and then it is isomorphic to but not equal to the external direct sum.
- Consider the subspaces $\mathbb{R} = \mathbb{R}\,1, \mathbb{R}\,x < \mathbb{R}[x]$. The sum is direct so the internal and external direct sum are isomorphic, but note the difference in the elements: