Permutations

Aim lecture: Intro algebra of permutations as a tool to study determinants.

Defn

Let $J_n = \{1, \ldots, n\}$.

- Let F_n the set of all functions of form $f : J_n \longrightarrow J_n$.
- We denote $f \in F_n$ by $[f(1)\ f(2)\ \ldots\ f(n)]$.
- A *permutation* is some $\sigma \in F_n$ which is invertible as a function i.e. it is bijective. The set of these is denoted S_n.
- $\sigma \in F_n$ is a permutation iff $[\sigma(1) \ldots \sigma(n)]$ "lists" the numbers $1, \ldots, n$ in some permuted order.
- For $\sigma \in S_n$, its inverse function $\sigma^{-1} \in S_n$.

e.g. $f = [1\ 2\ 2\ 4]$ is not a permutation but $\sigma = [3\ 2\ 1\ 4]$ is.
$\sigma(1) =$
$\sigma^{-1} = [$

Product structure

Facts

- For $f, g \in F_n$ we can define the composite function $fg : J_n \longrightarrow J_n \longrightarrow J_n$ (which is an elt of F_n) by $(fg)(i) = f(g(i))$.
- Composition of functions is associative so $(fg)h = f(gh)$ for $f, g, h \in F_n$.
- For $\sigma, \tau \in S_n$, $\sigma\,\tau \in S_n$ since $(\sigma\,\tau)^{-1} = \tau^{-1}\,\sigma^{-1}$. Why?
- The identity function $\mathrm{id} = [1\ 2\ \ldots\ n]$ is in S_n.
- For $f \in F_n$, $\mathrm{id}\, f = f = f\, \mathrm{id}$.

E.g. Let $\sigma = [2\ 1\ 3\ 4], \tau = [2\ 3\ 4\ 1]$
$\sigma\,\tau =$
$\sigma^2 =$

2-cycles

Defn

- For distinct $i, j \in J_n$, we define the permutn $\sigma = (i\ j)$ by $\sigma(i) = j, \sigma(j) = i, \sigma(k) = k$ if $k \neq i, j$. Any such permutn is called a *2-cycle*.
- $(i\ j)$ "swaps" i and j so $(i\ j)^2 = \text{id}$ & $(i\ j)^{-1} = (i\ j)$.

Prop

Any permutn is a product of 2-cycles since any permutation can be realised as a composite of swaps. Alternatively, observe algorithm below always works.

E.g. Write $\sigma = [3\ 2\ 4\ 1]$ as a product of 2-cycles.
Use fact: right multn by $(i\ j)$ swaps i-th & j-th entries in $[\sigma(1) \dots \sigma(n)]$.
$\sigma(1\ 4) =$

Matrix notation

Below, $\mathbb{F}$ will denote any set. From lecture 4 onwards however, we will reserve this symbol for a "field" such as $\mathbb{R}$ or $\mathbb{C}$.

Notn

Let $M_{mn}(\mathbb{F})$ denote the set of $m \times n$-matrices with entries in $\mathbb{F}$. Elements are often denoted by $(a_{ij})_{ij}$ (or just (a_{ij})) where the (i,j)-th entry is $a_{ij} \in \mathbb{F}$. The "outside" i, j tell you that the row index is i and the column index is j.

e.g. Let $(a_{ij})_{ij} = \left(\begin{smallmatrix} 1 & 2 \\ 3 & 4 \end{smallmatrix}\right)$.
$(a_{ji})_{ij} =$

Permuting rows & columns of matrices

Fact

Let $A = (a_{ij}) \in M_{mn}(\mathbb{F}), \sigma \in S_n, \tau \in S_m$.

- $(a_{i\,\sigma(j)})_{ij}$ represents the matrix A with the columns permuted according to σ^{-1}.
- $(a_{\tau(i)j})_{ij}$ represents the matrix A with the rows permuted according to τ^{-1}.

e.g. $A = (a_{ij}) = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$.