THE UNIVERSITY OF NEW SOUTH WALES

SCHOOL OF MATHEMATICS AND STATISTICS

November 2012

# MATH2601
# Higher Linear Algebra

(1) TIME ALLOWED – Three (3) hours

(2) TOTAL NUMBER OF QUESTIONS – 6

(3) ANSWER ALL QUESTIONS

(4) THE QUESTIONS ARE OF EQUAL VALUE

(5) ANSWER **EACH** QUESTION IN A **SEPARATE** BOOK

(6) THIS PAPER MAY BE RETAINED BY THE CANDIDATE

(7) **ONLY** CALCULATORS WITH AN AFFIXED “UNSW APPROVED” STICKER MAY BE USED

All answers must be written in ink. Except where they are expressly required pencils may only be used for drawing, sketching or graphical work.

**1.** **[16 marks]** For each of the following, say whether the statement is true or false and give a brief reason. You will get one mark for a correct true/false answer, and if your true/false answer is correct then you will get one mark for a good reason.

a) Consider the permutations $\sigma = [2\ 3\ 1], \tau = [2\ 1\ 3] \in S_3$. Then $\sigma\tau = [1\ 3\ 2]$.

b) There exists a matrix $A \in M_{33}(\mathbb{Z})$ with determinant 2 such that

$$A \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 4 \\ -8 \\ 16 \end{pmatrix}.$$

c) The following matrices are similar

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, B = \begin{pmatrix} 2 & 5 \\ 1 & 2 \end{pmatrix}.$$

d) In the vector space $\mathbb{R}[x]$, the sum $\mathbb{R}[x]_{\leq 1} + \mathbb{R}x^2$ is direct.

e) Let $V$ be a vector space complement to $\mathbb{R}(3 + x^2) + \mathbb{R}(1 + x^4)$ in $\mathbb{R}[x]_{\leq 5}$. Then $\dim V = 3$.

f) The linear map defined by the row matrix $(1 \quad x \quad x + x^2) : \mathbb{R}^3 \longrightarrow \mathbb{R}[x]_{\leq 2}$ defines a co-ordinate system on $\mathbb{R}[x]_{\leq 2}$.

g) The map $T : \mathbb{R}[x] \oplus \mathbb{R}[x] \longrightarrow \mathbb{R}[x] \oplus \mathbb{R}$ defined by

$$T \begin{pmatrix} f(x) \\ g(x) \end{pmatrix} = \begin{pmatrix} f(x) + 2g(x) \\ f(2) + g(1) \end{pmatrix}$$

is linear.

h) Suppose given 6 matrices in $M_{66}(\mathbb{C})$ with square zero i.e. $A^2 = 0$. Then at least two of the matrices must be similar to each other.

**2. [16 marks]**

a) Let $T : \mathbb{R}[x]_{\leq 3} \longrightarrow \mathbb{R}[x]_{\leq 2}$ be the map defined by the differential operator $T = (x-1)\frac{d^2}{dx^2} - 2\frac{d}{dx}$.

i) Explain briefly why $T$ is a linear map.

ii) Find the matrix representing $T$ with respect to the standard coordinate systems $(1\ x\ x^2\ x^3) : \mathbb{R}^4 \longrightarrow \mathbb{R}[x]_{\leq 3}$ and $(1\ x\ x^2) : \mathbb{R}^3 \longrightarrow \mathbb{R}[x]_{\leq 2}$.

iii) HENCE calculate a basis for the kernel of $T$.

iv) Find a basis for the image of $T$.

v) What are the rank and nullity of $T$?

b) Let $A = (-2\ \ -1\ \ 1\ \ 2) \in M_{14}(\mathbb{R})$ and $\mathbf{v} = (1, -1, 1, 0)^T$.

i) Show that $\mathbb{R}\mathbf{v} \subseteq \ker A$.

ii) Find a basis for a vector space complement to $\mathbb{R}\mathbf{v}$ in $\ker A$.

c) Let $U, V$ be subspaces of a finite dimensional vector space $W$ over a field $\mathbb{F}$. By considering the linear map $(\mathrm{id}|_U\ \ \mathrm{id}|_V) : U \oplus V \longrightarrow W$, show that $\dim(U+V) \leq \dim U + \dim V$.

**3. [16 marks]**

a) Consider the following complex matrix

$$A = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 3 & 0 \\ 1 & -1 & 2 \end{pmatrix}.$$

i) Show that $\left(\begin{smallmatrix} 1 \\ 1 \\ 1 \end{smallmatrix}\right)$ is an eigenvector for $A$.

ii) Determine the eigenvalues of $A$ and their algebraic multiplicities.

iii) For each eigenvalue, determine a basis for the corresponding eigenspace.

iv) For each eigenvalue, draw the corresponding Jordan form tableau.

v) Write down a Jordan canonical form $J$ for $A$ as well as the change of co-ordinates matrix $C$ that gives $J = C^{-1}AC$.

vi) Compute $\exp(tJ)$.

vii) Solve the differential equation

$$\frac{d\mathbf{y}}{dt} = A\mathbf{y} + e^{5t}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{y}(0) = \mathbf{0}.$$

Hint: You do not need to compute the exponential matrix in this case.

b) Let $A \in M_{nn}(\mathbb{C})$ be a matrix with Jordan canonical form

$$J = J_{n_1}(\lambda_1) \oplus \ldots \oplus J_{n_r}(\lambda_r).$$

Find a Jordan canonical form for the $2n \times 2n$-matrix

$$\begin{pmatrix} 0 & 2A \\ -A & 3A \end{pmatrix}.$$

Make sure you justify your answer fully.

**4. [16 marks]**

a) Consider the following subspace of $\mathbb{R}^3$

$$W = \mathbb{R}\begin{pmatrix}1\\2\\2\end{pmatrix} + \mathbb{R}\begin{pmatrix}5\\4\\-2\end{pmatrix}.$$

i) Apply the Gram-Schmidt algorithm to find an orthonormal basis for $W$.

ii) Write down the corresponding $QR$-factorisation of the matrix

$$A = \begin{pmatrix}1 & 5\\2 & 4\\2 & -2\end{pmatrix}.$$

iii) Find the orthogonal projection of $\left(\begin{smallmatrix}0\\0\\1\end{smallmatrix}\right)$ onto $W$.

iv) Find an orthogonal change of co-ordinates matrix $C \in GL_3(\mathbb{R})$ and a matrix $R \in M_{33}(\mathbb{R})$ such that $CRC^{-1}$ is the rotation matrix about the axis $W^\perp$ through an angle $\pi$.

b) Let $V = \mathbb{R}[x]_{\leq 1}$ with inner product $(f|g) = \int_0^1 f(t)g(t)dt$. Find $f \in V$ such that $(f|\cdot) = \frac{d}{dx} : V \longrightarrow \mathbb{R}$.

c) Let $\zeta = e^{2\pi i/3}$. For a subspace $V \leq \mathbb{C}^n$, we define the *$\zeta$-flection about $V$* to be the linear map $T = \zeta \mathrm{id} + (1-\zeta)P : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ where $P : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ is orthogonal projection onto $V$.

i) Show that $T^3 = \mathrm{id}$.

ii) Determine with reasons, the eigenspaces of $T$.

**5. [16 marks]**

a) Consider the quadratic form $Q(\binom{x_1}{x_2}) = 7x_1^2 - 8x_1x_2 + 13x_2^2$.

i) Find a symmetric matrix $A$ such that

$$Q(\left(\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right)) = (x_1 \quad x_2) A \left(\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right).$$

ii) Orthogonally diagonalise $A$.

iii) HENCE determine the points of the conic

$$7x_1^2 - 8x_1x_2 + 13x_2^2 = 1$$

which are closest to $(0, 0)$.

b) Let

$$A = \begin{pmatrix} 3i & 1+i \\ -1+i & 5i \end{pmatrix}.$$

i) Write down $A^*$.

ii) Determine with reasons if $A$ can be unitarily diagonalised.

c) Let $\zeta = e^{2\pi i/3}$ and $A \in M_{33}(\mathbb{C})$ be a matrix with the following eigenspaces.

$$E_1 = \mathbb{C}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \quad E_\zeta = \mathbb{C}\begin{pmatrix} 1 \\ 1 \\ i \end{pmatrix}, \quad E_{\zeta^2} = \mathbb{C}\begin{pmatrix} 1 \\ 1 \\ -2i \end{pmatrix}$$

i) What is $\det(A)$?

ii) Is $A$ a unitary matrix? Justify your answer fully.

iii) Is $A$ a rotation matrix? If so, determine the angle of rotation. Otherwise, fully justify why the matrix is not a rotation matrix.

**6. [16 marks]**

a) Let $\mathbb{F}$ be a field and $SL_2(\mathbb{F}) = \{A \in M_{22}(\mathbb{F}) | \det(A) = 1\}$. Show that matrix multiplication restricts to a binary operation on $SL_2(\mathbb{F})$ and that $SL_2(\mathbb{F})$ is a group under this binary operation.

b) Let $A \in M_{44}(\mathbb{C})$ be a matrix with characteristic polynomial $\text{cp}_A(\lambda) = \lambda^4 + \lambda^3 - 2\lambda + 3$.

   i) Find $\det(A)$ and $\text{tr}(A)$.

   ii) Express $A^{-1}$ as a linear combination of $I, A, A^2, A^3$. State the names of any theorems you use.

c) Let $T : V \longrightarrow V$ be a linear map and $V', V'' \leq V$ be $T$-invariant subspaces. Prove that $V' + V''$ is a $T$-invariant subspace of $V$.

d) Let $V$ be a vector space and $W$ be a subspace. Let

$$M = \{T \in L(V, V) | \;\; W \text{ is } T\text{-invariant}\}.$$

Prove that $M$ is a subspace of $L(V, V)$.

e) Let $V$ be a finite dimensional $\mathbb{C}$-space and $S, T : V \longrightarrow V$ be linear maps such that the following hold: $S^3 = 2\text{id}, S \circ T = e^{2\pi i/3} T \circ S$.

   i) Show that if $T$ is invertible, then $\dim V$ is a multiple of 3.

   ii) Show that if $\dim V \leq 3$ then $\text{tr}(T) = 0$.