# School of Mathematics and Statistics, UNSW
# MATH2601 Higher Linear Algebra, Semester 2, 2012
# Class Test 2 Version A

Calculators are not permitted. Time allowed: 40 minutes.

This sheet must be filled in and stapled to the front of your answers

| Student's Surname Initials | Tutor | Student Number |
|---|---|---|
| | | |

Throughout, as in lectures, $\mathbb{F}$ always denotes a field.

(1) [6 marks] For each of the following, say whether the statement is true or false and give a brief reason. You will get one mark for a correct true/false answer, and if your true/false answer is correct then you will get one mark for a good reason.

(i) Let $V$ be a vector space complement to $\mathbb{R}(1-x+x^2)$ in $\mathbb{R}[x]_{\le 3}$. Then $\dim V = 2$.

(ii) Let $A \in M_{33}(\mathbb{R})$ be a matrix with

$$\ker(A-I) = \mathbb{R}(1,0,2)^T + \mathbb{R}(3,1,1)^T, \ker(A-2I) = \mathbb{R}(4,1,1)^T.$$

Then $A$ is diagonalisable.

(iii) Let

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \in M_{22}(\mathbb{R}).$$

Then there are infinitely many $A$-invariant subspaces of $\mathbb{R}^2$.

(2) [4 marks]

Consider the discrete time system $\mathbf{v}(k+1) = A\mathbf{v}(k)$ where $A \in M_{33}(\mathbb{R})$ has Jordan canonical form $C^{-1}AC = J = J_2(3) \oplus J_1(1)$. Solve for $\mathbf{v}(k)$ if

$$C = \begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{v}(0) = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

(3) [2 marks]

Consider the co-ordinate systems $C_1 = (1\ x) : \mathbb{R}^2 \longrightarrow \mathbb{R}[x]_{\le 1},\ C_2 = (1\ x\ x^2) : \mathbb{R}^3 \longrightarrow \mathbb{R}[x]_{\le 2}$. Let $T : \mathbb{R}[x]_{\le 2} \longrightarrow \mathbb{R}[x]_{\le 1}$ be the linear map given by the matrix

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \end{pmatrix}$$

with respect to the co-ordinate systems $C_2, C_1$. Find a basis for $\ker T$.

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(4) [7 marks]

Consider the following matrix

$$A = \begin{pmatrix} 3 & 2 \\ -\frac{1}{2} & 1 \end{pmatrix}.$$

(i) Find the eigenvalues of $A$.
(ii) Write down the algebraic and geometric multiplicities of the eigenvalues.
(iii) Find a Jordan canonical form for $A$.
(iv) Write down a change of co-ordinates matrix $C$ such that $J = C^{-1}AC$.

(5) [2 marks]

Let $T, S : V \longrightarrow V$ be linear maps. Let $W \leq V$ be a subspace which is both $T$-invariant and $S$-invariant. Prove that $W$ is $S \circ T$-invariant. Make sure your answer is set out logically and your reasoning is complete.

# School of Mathematics and Statistics, UNSW
# MATH2601 Higher Linear Algebra, Semester 2, 2012
# Class Test 2 Version B

Calculators are not permitted. Time allowed: 40 minutes.

This sheet must be filled in and stapled to the front of your answers

| Student's Surname Initials | Tutor | Student Number |
|---|---|---|
| | | |

Throughout, as in lectures, $\mathbb{F}$ always denotes a field.

(1) [6 marks] For each of the following, say whether the statement is true or false and give a brief reason. You will get one mark for a correct true/false answer, and if your true/false answer is correct then you will get one mark for a good reason.

(i) Let $T : M_{23}(\mathbb{F}) \longrightarrow M_{22}(\mathbb{F})$ be a surjective linear map. Then the nullity of $T$ is 2.

(ii) The following matrices are similar
$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 3 & 5 \end{pmatrix}.$$

(iii) Let $A \in M_{33}(\mathbb{R})$ be a matrix with
$$\ker(A - I) = \mathbb{R}(1,0,2)^T + \mathbb{R}(8,0,1)^T, \ker(A - 2I) = \mathbb{R}(4,1,1)^T.$$
Then $A$ is diagonalisable.

(2) [2 marks]

Consider the co-ordinate systems $C_1 = (1\ x) : \mathbb{R}^2 \longrightarrow \mathbb{R}[x]_{\leq 1}$, $C_2 = (1\ x\ x^2) : \mathbb{R}^3 \longrightarrow \mathbb{R}[x]_{\leq 2}$. Let $T : \mathbb{R}[x]_{\leq 2} \longrightarrow \mathbb{R}[x]_{\leq 1}$ be the linear map given by the matrix
$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 3 \end{pmatrix}$$
with respect to the co-ordinate systems $C_2, C_1$. Find a basis for $\ker T$.

(3) [4 marks] Solve the inital value problem $\frac{d\mathbf{y}}{dt} = A\mathbf{y}(t), \quad \mathbf{y}(0) = (1,0,1)^T$ where $A \in M_{33}(\mathbb{R})$ has Jordan canonical form $C^{-1}AC = J_2(3) \oplus J_1(1)$ and
$$C = \begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

PLEASE TURN OVER

(4) [7 marks]

Consider the matrix

$$A = \begin{pmatrix} 3 & 1 & -1 \\ 0 & 4 & -1 \\ 0 & 1 & 2 \end{pmatrix}.$$

(i) Find a basis for the eigenspace $E_3$ corresponding to the eigenvalue 3.

(ii) Show that the generalised eigenspace $E_3(\infty) = \mathbb{C}^3$.

(iii) Write down a Jordan canonical form $J$ for $A$.

(iv) Write down a change of co-ordinates matrix $C$ such that $J = C^{-1}AC$.

(5) [2 marks]

Let $T : V \longrightarrow V$ be a linear map and $W, W' \leq V$ be $T$-invariant subspaces. Prove that $W \cap W'$ is also $T$-invariant. Make sure your answer is set out logically and your reasoning is complete.