School of Mathematics and Statistics, UNSW
MATH2601 Higher Linear Algebra, Semester 2, 2012
Class Test 1 Version A

Calculators are not permitted. Time allowed: 40 minutes.

This sheet must be filled in and stapled to the front of your answers

Student's Surname Initials	Tutor	Student Number

Throughout, as in lectures, $\mathbb{F}$ always denotes a field. Recall also that $\text{ev}_x(f) = f(x)$.

(1) [12 marks] For each of the following, say whether the statement is true or false and give a brief reason. You will get one mark for a correct true/false answer, and if your true/false answer is correct then you will get one mark for a good reason.

(i) Consider the permutation $\sigma = [2\ 3\ 1]$. Then $\sigma^2 = \text{id}$.

(ii) Let $R : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ denote some reflection about a line L in $\mathbb{R}^2$ of the form $L = \mathbb{R}\mathbf{u}$. Then $R^4 = \text{id}$.

(iii) The row matrix $(x^2 \quad x \quad 1) : \mathbb{R}^3 \longrightarrow \mathbb{R}[x]_{\leq 2}$ defines a co-ordinate system on $\mathbb{R}[x]_{\leq 2}$.

(iv) Consider the linear maps $\text{ev}_0, \text{ev}_1 \in L(\mathbb{C}[x], \mathbb{C})$. Then ev_0, ev_1 are scalar multiples of each other.

(v) Let $T : V \longrightarrow W$ be an isomorphism of vector spaces V, W. Then the linear map $S : V^2 \longrightarrow W^2$ defined by the matrix

$$\begin{pmatrix} T & T \\ T & T \end{pmatrix}$$

is also an isomorphism of vector spaces.

(vi) Let G be a group. Then there is an action of G on G defined by $g.h = hg^{-1}$ for any $g, h \in G$.

(2) [4 marks]

Consider the following subspaces of $\mathbb{C}^3$: $W = \mathbb{C}(1, -2, 1, 0)^T + \mathbb{C}(0, 1, 0, -1,)^T, W' = \mathbb{C}(1, 1, 0, -2)^T$. Justify your answers below with appropriate computations.

(i) Is $(1, 1, 1, 3)^T \in W + W'$?

(ii) Is the sum $W + W'$ is direct?

(3) [4 marks]

(i) For any permutation $\sigma \in S_n$, show that $\text{sgn}(\sigma^{-1}) = \text{sgn}(\sigma)$.

(ii) Use the definition of the determinant to show that $\det(A^T) = \det(A)$ for any matrix $A \in M_{nn}(\mathbb{F})$.

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(4) [4 marks]

In this question, we consider the map $T : \mathbb{R}[x]_{\leq 1} \oplus \mathbb{R} \longrightarrow \mathbb{R}^3$ defined by

$$T\begin{pmatrix} f(x) \\ \beta \end{pmatrix} = \begin{pmatrix} f(0)+\beta \\ f'(0)-\beta \\ \beta \end{pmatrix}.$$

(i) Show T is linear by finding some matrix of linear maps which represents it.

(ii) Let $S : \mathbb{R}^3 \longrightarrow \mathbb{R}[x]_{\leq 1} \oplus \mathbb{R}$ be the linear map given by the matrix

$$\begin{pmatrix} 1 & x & x-1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Find $S(1,2,2)^T$.

(iii) Show that $S \circ T = \text{id}$. Justify your answer fully. Hint: for any $f(x) \in \mathbb{R}[x]_{\leq 1}, \beta \in \mathbb{R}$ you need to show $(S \circ T)\binom{f(x)}{\beta} = \binom{f(x)}{\beta}$.

(5) [2 marks]

Let $T : V \longrightarrow W$ be a bijective linear map. Prove from the definition of linearity that T^{-1} is also linear.

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Class Test 1 Version B

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Student's Surname Initials	Tutor	Student Number

Throughout, as in lectures, $\mathbb{F}$ always denotes a field. Recall also that $\mathrm{ev}_x(f) = f(x)$.

(1) [12 marks] For each of the following, say whether the statement is true or false and give a brief reason. You will get one mark for a correct true/false answer, and if your true/false answer is correct then you will get one mark for a good reason.

(i) Consider the permutation $\sigma = [2\ 3\ 1\ 4]$. Then $\sigma^{-1} = [1\ 3\ 2\ 4]$.
(ii) The group identity in S_3 is $[1\ 2\ 3\ 4]$.
(iii) Consider the $\mathbb{C}$-space $V = L(\mathbb{C}^2, \mathbb{C})$ and the subset W of linear maps $T \in V$ which satisfy $T\binom{1}{2} = 0$. Then W is a subspace of V.
(iv) The row matrix $(x^2 - 1\ \ x\ \ 1) : \mathbb{R}^3 \longrightarrow \mathbb{R}[x]_{\leq 2}$ defines a co-ordinate system on $\mathbb{R}[x]_{\leq 2}$.
(v) In the $\mathbb{R}$-space $V = C^\infty(\mathbb{R}) \oplus \mathbb{R}$, $\binom{\cos(x+\pi/4)}{0} \in V$ is a linear combination of $\binom{\cos x}{1}, \binom{\sin x}{1}$.
(vi) The linear map $T : \mathbb{R}[x] \longrightarrow \mathbb{R}[x] \oplus \mathbb{R}$ given by the matrix

$$\begin{pmatrix} \frac{d}{dx} \\ \mathrm{ev}_0 \end{pmatrix}$$

is an isomorphism of vector spaces.

(2) [4 marks]
Consider the subspace $W = \mathbb{R}(2 - x^2) + \mathbb{R}(x + 2x^2)$ of $\mathbb{R}[x]$.
(i) Is the sum $\mathbb{R}[x]_{\leq 1} + W$ direct? Justify your answer fully.
(ii) Show that $\mathbb{R}[x]_{\leq 1} + W = \mathbb{R}[x]_{\leq 2}$.

(3) [4 marks]
Let $P : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be orthogonal projection onto $\mathbf{u} = \frac{1}{5}\binom{4}{3}$.
(i) Find the matrix representing P.
(ii) Let L be the line in $\mathbb{R}^2$ through $\binom{0}{0}$ perpendicular to $\mathbf{u}$. If R is reflection about L, write R as a linear combination of P and the identity map.
(iii) Hence or otherwise find the matrix representing R.

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(4) [4 marks]

(i) For any permutation $\sigma \in S_n$, show that $\operatorname{sgn}(\sigma^{-1}) = \operatorname{sgn}(\sigma)$.

(ii) Use the definition of the determinant to show that $\det(A^T) = \det(A)$ for any matrix $A \in M_{nn}(\mathbb{F})$.

(5) [2 marks]

Let $V', V'', W \leq V$. Prove that $(V' \cap W) + (V'' \cap W) \subseteq (V' + V'') \cap W$.