Eigenvectors

**Aim lecture:** The simplest $T$-invariant subspaces are 1-dim & these give rise to the theory of eigenvectors. To compute these we introduce the similarity invariant, the characteristic polynomial.

**Prop-Defn**

Let $T : V \rightarrow V$ be linear. The following are equivalent condns on $v \in V$.

1. $Fv$ is a $T$-invariant (automatically 1-dimensional) subspace.
2. $Tv = \lambda v$ for some $\lambda \in F$.
3. $v \in \ker(T - \lambda I)$ for some $\lambda \in F$.

If these hold & furthermore $v \neq 0$ we say $v$ is an *eigenvector* for $T$ with *eigenvalue* $\lambda$.

**Proof.** Clearly 2) $\iff$ 3) since

$$Tv = \lambda v \iff Tv - \lambda id v = 0 \iff v \in \ker(T - \lambda I).$$

2) $\implies$ 1) by lemma on testing invariance, lecture 20.

For 1) $\implies$ 2) note
Eigenvalues & eigenspaces of an endomorphism

Defn
Let $T : V \to V$ be linear. An eigenvalue of $T$ is a scalar $\lambda \in F$ such that there is an e-vector $v$ for $T$ with eigenvalue $\lambda$. Given such an e-value, the $\lambda$-eigenspace of $T$ is the subspace $E_\lambda = \ker(T - \lambda \text{id}) \leq V$. The geometric multiplicity of $\lambda$ is $\dim E_\lambda$.

E.g. Any real number $\lambda$ is an e-value for $T = \frac{d}{dx} : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$ since
To find e-values of square matrices we need

**Prop-Defn**

Let \( A = (a_{ij}) \in M_{nn}(\mathbb{F}) \). The *characteristic polynomial of \( A \) is the function* 
\[ cp_A(\lambda) = \det(A - \lambda I_n) \]
where \( I_n \) is the \( n \times n \)-identity matrix. This function is a polynomial function of degree \( n \) with coefficients in \( \mathbb{F} \).

**Proof.** Note that 
\[ A - \lambda I_n = (a'_{ij}) \]
where \( a'_{ij} = a_{ij} \) if \( i \neq j \) whilst \( a'_{ii} = a_{ii} - \lambda \). Now

\[
\det(A - \lambda I_n) = \sum_{\sigma \in S_n} \text{sgn}(\sigma) \prod_{i=1}^{n} a'_{i \sigma(i)}
\]

which is clearly a polynomial function of \( \lambda \).

The summand corresponding to \( \sigma = \text{id} \) is 
\[ (a_{11} - \lambda)(a_{22} - \lambda) \ldots (a_{nn} - \lambda) \]
which has degree \( n \).

Any other summand contains at least two non-diagonal entries so has degree \( \leq n - 2 \). Hence, \( \deg cp_A(\lambda) = n \).

**Scholium** The co-efficient of \( \lambda^{n-1} \) in \( cp_A(\lambda) \) is \((-1)^{n-1} \sum_i a_{ii}\).
Relation with e-values. Algebraic multiplicity

Prop-defn
Let $A \in M_{nn}(\mathbb{F})$. Then $\lambda$ is an e-value for $A$ iff it is a root of the characteristic polynomial of $A$. In this case, the multiplicity of the root is called the algebraic multiplicity of the e-value.

Proof. $\lambda$ is an e-value iff $A$ has an e-vector with e-value $\lambda$ iff $\ker(A - \lambda I_n) \neq \mathbf{0}$ iff $A - \lambda I_n$ is not invertible iff $\det(A - \lambda I_n) = 0$.

E.g. Find the e-values of $A = \begin{pmatrix} 3 & 1 \\ -1 & 1 \end{pmatrix}$ & their algebraic & geometric multiplicities.
Two matrices $A, B \in M_{nn}(\mathbb{F})$ are similar if there exists $C \in GL_n(\mathbb{F})$ such that $A = C^{-1}BC$ i.e. $A$ is a matrix representing $B$ wrt some co-ordinate system. Being similar is an equivalence relation. The set of all matrices similar to $A$ is called the similarity class of $A$.

**Proof.** Omitted.

**E.g.** $A = \begin{pmatrix} 3 & 1 \\ -1 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix}$ are not similar for if $A = C^{-1}BC$ then
Similarity invariants

**Defn**

A function of the form \( f : M_{nn}(\mathbb{F}) \to X \) is a **similarity invariant** if \( f(A) = f(B) \) whenever \( A, B \) are similar.

**E.g. 1** The characteristic polynomial \( \text{cp} : M_{nn}(\mathbb{F}) \to \mathbb{F}[\lambda] : A \mapsto \text{cp}_A(\lambda) \) is a similarity invariant since if \( B = C^{-1}AC \) for some \( C \in GL_n(\mathbb{F}) \) then

\[
\det(B - \lambda I_n) = \det(C^{-1}AC - \lambda I_n) = \det(C^{-1}[A - \lambda I_n]C) = \]

**E.g. 2** In particular, any of the co-efficients of the characteristic polynomial are similarity invariants. The important ones (up to sign) are the determinant \( \det(A) = \text{cp}_A(0) \) & the **trace** which is defined to be \( \text{tr}(A) = \sum_{i=1}^{n} a_{ii} \) where \( A = (a_{ij}) \).

**E.g. 3** Similarly, the set of eigenvalues is a similarity invariant.
Characteristic polynomials of endomorphisms

Similarity invariants can be extended to endomorphisms of finite dimensional vector spaces. For example

**Prop-Defn**

Let $V = \text{fin dim } \mathbb{F}$-space & $T : V \rightarrow V$ be linear. For any co-ordinate system $C : \mathbb{F}^n \rightarrow V$, we may define the *characteristic polynomial* of $T$, denoted $\text{cp}_T(\lambda)$, to be the characteristic polynomial of the representing matrix $C^{-1} \circ T \circ C$. This is well-defined since given any other co-ordinate system $C_1 : \mathbb{F}^n \rightarrow V$, the characteristic polynomials of $C^{-1} \circ T \circ C$ & $C_1^{-1} \circ T \circ C_1$ are the same, so the definition is independent of the choice of co-ord system.

**Proof.** We need only check equality of characteristic polynomials by showing $C^{-1} \circ T \circ C$ & $C_1^{-1} \circ T \circ C_1$ are similar. Indeed

$$C_1^{-1} \circ T \circ C_1$$

**Rem** We similarly can define $\text{det}(T)$, $\text{tr}(T)$ etc.

**E.g.** We have seen that $T = \frac{d}{dx} : \mathbb{R} \cos x \oplus \mathbb{R} \sin x \rightarrow \mathbb{R} \cos x \oplus \mathbb{R} \sin x$ is represented by the matrix
Computing e-values & e-vectors of endomorphisms

Using co-ordinates, we can calculate e-vectors, e-values & e-spaces using

**Prop**

Let \( V = \text{fin dim } \mathbb{F}-\text{space} \) & \( T : V \rightarrow V \) be linear. Let \( C : \mathbb{F}^n \rightarrow V \) be a co-ord system & \( A = C^{-1} \circ T \circ C \) be the representing matrix. Then

1. \( x \in \mathbb{F}^n \) is an e-vector of \( A \) with e-value \( \lambda \) iff \( Cx \) is an e-vector of \( T \) with e-value \( \lambda \).

2. The e-values of \( T \) & \( A \) are the same. They are the roots of \( \text{cp}_A(\lambda) = \text{cp}_T(\lambda) \).

3. If \( E_\lambda \) is the \( \lambda \)-e-space of \( A \) then \( C(E_\lambda) \) is the \( \lambda \)-e-space of \( T \).

**Proof.** We just prove 1), as 2) & 3) readily then follow.

\[ Ax = \lambda x \iff C^{-1} \circ T \circ C x = \lambda x \iff \]
Example

E.g. We compute the e-vectors & e-values of $T : \mathbb{R}[x]_{\leq 2} \rightarrow \mathbb{R}[x]_{\leq 2}$ defined by

$$(Tp)(x) = xp'(x) - 2p'(x) - p(x).$$