School of Mathematics, UNSW

MATH5765: Homology, Problem Sheet 3 & Exam Prep

The exam will be a mixture of "bookwork" that is material from the lectures, and questions similar if not identical to those on the previous two problem sheets and this one. I will assume you know all the results in your assignment questions and am happy for you to use them in the exam. If you understand the lecture material and can do all the problem sheet questions, you should get at least 90%.

From the lectures, I'd like you to understand the statements of all results but not necessarily the proofs. Certainly the vague proof sketches I gave on topological/homotopy invariance (lectures 10-13) are not examinable. For example, I don't expect you to know the proof of the simplicial approximation theorem, just to understand the role it plays in allowing one to prove functoriality of the homology functor. A lot of the proof sketches are for your information only.

1. What's $H_p(S^n;\mathbb{Q})$?

2. Compute $\mathbb{Z}/10\mathbb{Z}\otimes_{\mathbb{Z}}\mathbb{Z}/6\mathbb{Z}$. Let $\phi:\mathbb{Z}/6\mathbb{Z}\longrightarrow\mathbb{Z}/6\mathbb{Z}$ be multiplication by the integer n. What is the map $\mathbb{Z}/10\mathbb{Z}\otimes_{\mathbb{Z}}\phi$? For what values of n is it zero?

3. Compute $\mathbb{Z}/2\mathbb{Z}\otimes_{\mathbb{Z}}(\oplus_{n\in\mathbb{N}}\mathbb{Z}/(2n+1)\mathbb{Z})$.

4. Consider the following exact sequence of abelian groups.
$$A\xrightarrow{\phi}B\longrightarrow\mathbb{Z}/n\mathbb{Z}\longrightarrow 0.$$
For what values of n is $\mathbb{Z}/2\mathbb{Z}\otimes_{\mathbb{Z}}\phi$ surjective?

5. Find two free resolutions of the $\mathbb{Z}$-module $\mathbb{Z}^2\oplus\mathbb{Z}/6\mathbb{Z}\oplus\mathbb{Z}/5\mathbb{Z}$.

6. For k a field, find a free resolution of the $k[x,y]$-module $k[x,y]/(x,y^2)$.

7. For k a field, we define a module M over $R=k[x,y]$ as follows. As a group, $M=k[z]\oplus k[z]$. Scalar multiplication is given by $f(x,y)(p(z),q(z)):=(f(z,0)p(z),f(0,z)q(z))$. Show that M is an R-module. Find a free resolution for M.

8. Let M be the ideal of $R=k[x,y,z]$ (k a field) generated by x,z. Find a free resolution for M.

9. Consider an exact sequence of R-modules $0\longrightarrow N\longrightarrow M\longrightarrow F\longrightarrow 0$ where F is free. Show that $M\simeq N\oplus F$.

10. Let $F:R-\text{Mod}\longrightarrow S-\text{Mod}$ be a right exact functor such that for some i, $L_iF=0$ i.e. for any $M\in R-\text{Mod}$ we have $(L_iF)M=0$. Show that $L_{i+1}F=0$ too.

11. Calculate $\mathrm{Tor}_i^{\mathbb{Z}}(\mathbb{Z}/8\mathbb{Z}, \mathbb{Z}/12\mathbb{Z})$.

12. Let $R = k[x, y]$ (as always, k is some field). Calculate and compare $\mathrm{Tor}_i^R(R/(x, y), R/(x))$ and $\mathrm{Tor}_i^R(R/(x), R/(x, y))$. How does your answer change when you consider Tor as a graded vector space?

13. Calculate $H_p(S^n; \mathbb{Z}/n\mathbb{Z})$.

14. For X the five-fold dunce's cap of Q5, problem sheet 1, calculate $H_p(X; \mathbb{Z}/5\mathbb{Z})$. Incidentally, it is an interesting exercise to compute the homology of the five-fold dunce's cap using the Mayer-Vietoris sequence.

15. Let $R = k[x, y, z]$ where k is a field. Consider a graded free resolution of the graded R-module M

$$0 \longrightarrow R[-3]^2 \longrightarrow R[-2]^3 \longrightarrow R \longrightarrow M \longrightarrow 0.$$

Find the Hilbert polynomial of M. Show that M is not isomorphic to a submodule of R. Compute $\mathrm{Tor}_i^R(k, M)$.

Hints: I will put the hints to all questions here and let you sort out which hints go with which questions.

The module M can be generated by two elements $(1, 0), (0, 1)$.

Use the Hom-lifting lemma of lecture 22 to show F is isomorphic to a submodule of M and then some module version of the "internal" characterisation of products of groups as in my Group Theory course (lecture 12 or prop 9.2 in Kenneth's notes on my webpage).

Consider an exact sequence of the form $0 \longrightarrow M' \longrightarrow P \longrightarrow M \longrightarrow 0$ where P is a free module. Apply the long exact sequence of the derived functors.