# School of Mathematics, UNSW

# MATH5765: Homology, Problem Sheet 2

The second assignment consists of questions 1,9,11,15.

1. For $A$ a subset of a topological space $X$, a retraction of $X$ to $A$ is a continuous map $r : X \longrightarrow A$ with $r(a) = a$ for all $a \in A$. Show there is no retraction of $X = S^2$ to its equator.

2. Show that any two maps $f, g : X \longrightarrow \mathbb{R}^n$ are homotopic.

3. Show that the real projective plane of problem sheet 1 is not homotopy equivalent to a point.

4. Of the letters below, which are homotopy equivalent?

   A B C D E F

5. Which of the following are homotopy equivalent: a)$\mathbb{R}^3$ - $z$-axis b) $S^1$ c) torus $\mathbb{T}^2$ - point?

6. If $A$ and $B$ are abelian groups, show that $(A \oplus B)_{\mathbb{Q}} \simeq A_{\mathbb{Q}} \oplus B_{\mathbb{Q}}$.

7. In this question, we view the 2 dimensional torus $\mathbb{T}^2$ as $S^1 \times S^1$. Let $\phi : \mathbb{T}^2 \longrightarrow \mathbb{T}^2$ be the continuous map which sends $(x, y) \mapsto (y, x)$. Compute the trace of $H_p(\phi)_{\mathbb{Q}} : H_p(\mathbb{T}^2)_{\mathbb{Q}} \longrightarrow H_p(\mathbb{T}^2)_{\mathbb{Q}}$.

8. In this question, we verify a special case of the Hopf trace formula explicitly. Calculate $\chi(S^n)$ from the definition via homology. Show that you get the same answer if you calculate it by triangulating $S^n$ with the proper faces of an $n+1$-simplex and taking the alternating sum of the number of $p$-simplices.

9. Consider an exact sequence of the form

$$0 \longrightarrow \mathbb{Z}^a \longrightarrow \mathbb{Z}^b \longrightarrow \mathbb{Z}^c \longrightarrow \mathbb{Z}^d \longrightarrow \mathbb{Z}^e \longrightarrow 0$$

   where $a, b, c, d, e \in \mathbb{N}$. Using the Hopf trace formula (and the functor $(-)_{\mathbb{Q}}$) or otherwise, find $a - b + c - d + e$.

10. Let $\phi : S^n \longrightarrow S^n$ be a continuous map where $n$ is odd. If $H_n(\phi) : H_n(S^n) \longrightarrow H_n(S^n)$ is not the identity homomorphism, show that $\phi$ has a fixed point.

11. Let $P^2$ be the real projective plane as defined in problem sheet 1. Show that any continous map $f : P^2 \longrightarrow P^2$ has a fixed point.

12. This question is an exercise in diagram chasing. Prove the (Steenrod) 5-lemma below. Consider a commutative diagram (in some module category) below with exact rows.

$$\begin{array}{ccccccccc} A_1 & \longrightarrow & A_2 & \longrightarrow & A_3 & \longrightarrow & A_4 & \longrightarrow & A_5 \\ \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_3} & & \downarrow{\scriptstyle f_4} & & \downarrow{\scriptstyle f_5} \\ B_1 & \longrightarrow & B_2 & \longrightarrow & B_3 & \longrightarrow & B_4 & \longrightarrow & B_5 \end{array}$$

Show that if $f_1, f_2, f_4, f_5$ are isomorphisms then so is $f_3$.

13. Let $K$ be a simplicial complex and $L$ be a subcomplex. Show that there is a long exact sequence

$$\ldots \longrightarrow H_p(L) \longrightarrow H_p(K) \longrightarrow H_p(K, L) \longrightarrow H_{p-1}(L) \longrightarrow \ldots$$

For the cone $w * K$, show that $H_p(w * K, K) \simeq \tilde{H}_{p-1}(K)$.

14. This question helps you fill in details to the application of Mayer-Vietoris in lecture 18. Let $I = [0, 1]$ and $X$ be a triangulable topological space i.e. $X \in \text{TrTop}$. Show that $X$ and $X \times I$ are homotopy equivalent. More precisely, if $f_t : X \longrightarrow X \times I : x \mapsto (x, t)$ for $t \in I$, show that $f_t$ is a homotopy equivalence. Moreover, show that $H_p(f_t)$ induce the same isomorphism for every $t$.

15. Compute $H_p(S^1 \times S^2)_{\mathbb{Q}}$. Hint: If $I = [0, 1]$ then $S^1 \times S^2$ is the union of 2 copies of $S^2 \times I$. Use the Mayer-Vietoris sequence as in lecture 18 and the previous question.

16. Let $K$ be a simplicial complex. We may assume that it is embedded in a sufficiently big Euclidean space so that there exist points $v, w$ such that the cones $|v * K|, |w * K|$ only intersect in $|K|$. We define the suspension of $K$ to be the simplicial complex

$$S(K) := v * K \cup w * K.$$

Show that $\tilde{H}_p(S(K)) \simeq \tilde{H}_{p-1}(K)$.