School of Mathematics, UNSW

MATH5765: Homology, Problem Sheet 1

The first assignment consists of questions 2,8,10 and 12 and is due by the end of week 6. Pictures are below.

1. Consider the "n-gon" simplex K which has vertices $\{a_i | i \in \mathbb{Z}/n\mathbb{Z}\}$ and the only other simplices are those of the form $a_i a_{i+1}$. Calculate $H_p(K)$ for all p.

2. Let K be the real projective plane as triangulated in the picture below. Compute its homology. (You should put about as much working as I did for the torus. The most important point is that you demonstrate that you can correctly adapt the argument there to this case.)

3. Let K be the Klein bottle as triangulated below. Compute its homology.

4. For which values of i do you know for sure that $H_i(K^{(p)}) \simeq H_i(K)$?

5. Triangulate the "5-fold dunce's cap" below and compute its homology.

6. Let A be any finitely generated abelian group. Find a simplicial complex K with $H_1(K) \simeq A$.

7. Describe the relative homology $H_0(K, K_0)$ in terms of the topology of $|K|, |K_0|$.

8. Let K be a simplicial complex such that $|K|$ has n connected components. Show that $\tilde{H}_0(K) \simeq \mathbb{Z}^{n-1}$ but $\tilde{H}_p(K) \simeq H_p(K)$ for $p \neq 0$.

9. Let K be the $2n$-gon simplicial complex of question 1. Let K_0 be the 0-dimensional subcomplex consisting of $\{a_0, a_n\}$. Compute $H_p(K, K_0)$.

10. Let Σ^{n-1} be the simplicial complex consisting of all proper faces of an n-simplex. Find all simplicial maps $f : \Sigma^{n-1} \longrightarrow \Sigma^{n-1}$ and the corresponding induced maps f_* on homology.

11. Let $\mathrm{Mor}(C_., C'_.)$ be the set of chain maps from $C_.$ to $C'_.$. Show that being homotopic is an equivalence relation on $\mathrm{Mor}(C_., C'_.)$. Given homotopic chain maps $f, f' \in \mathrm{Mor}(C_., C'_.)$ and another chain map $g : C'_. \longrightarrow C''_.$, show that $g \circ f$ is homotopic to $g \circ f'$.

12. For $n \in \mathbb{N}$, consider the chain complexes

$$C_. : 0 \longrightarrow \mathbb{Z} \xrightarrow{n^2} \mathbb{Z} \longrightarrow 0 \qquad\qquad C'_. : 0 \longrightarrow \mathbb{Z} \xrightarrow{n} \mathbb{Z} \longrightarrow 0$$

where the superscript n or n^2 denotes multiplication by that integer. Find all chain maps $C_. \longrightarrow C'_.$. What are the homotopy equivalence classes in $\mathrm{Mor}(C_., C'_.)$?

13. Let K be the 3-gon simplicial complex with vertices p, q, r and L be the torus with the usual triangulation. We have 2 simplicial maps $\phi, \phi' : K \longrightarrow L$ which send p, q, r to b, f, i and c, g, j respectively. Clearly ϕ, ϕ' are not contiguous but do induce the same map on homology. Find an acyclic carrier Φ from K to L such that both ϕ, ϕ' are carried by Φ.