HOMOLOGY AND HOMOLOGICAL ALGEBRA, D. CHAN

1. Simplicial complexes

Motivating question for algebraic topology: how to tell apart two topological spaces? One possible solution is to find distinguishing features, or invariants. These will be homology groups. How do we build topological spaces and record on computer (that is, finite set of data)?

Definition 1.1. *Let $a_0, \dots, a_n \in \mathbb{R}^N$. The* ***span*** *of $a_0, \dots, a_n$ is*

$$\begin{aligned} a_0 \dots a_n &:= \left\{ \sum_{i=0}^{n} \lambda_i a_i \mid \lambda_i \geqslant 0, \lambda_1 + \dots + \lambda_n = 1 \right\} \\ &= \text{convex hull of } \{a_0, \dots, a_n\}. \end{aligned}$$

The points $a_0, \dots, a_n$ are ***geometrically independent*** *if $a_1 - a_0, \dots, a_n - a_0$ is a linearly independent set over $\mathbb{R}$. Note that this is independent of the order of $a_0, \dots, a_n$. In this case, we say that the simplex $a_0 \dots a_n$ is n* ***-dimensional****, or an n* ***-simplex****. Given a point $\sum_{i=1}^n \lambda_i a_i$ belonging to an n-simplex, we say it has* ***barycentric coordinates*** *$(\lambda_0, \dots, \lambda_n)$. One can use geometric independence to show that this is well defined.*
A (proper) ***face*** *of a simplex $\sigma = a_0 \dots a_n$ is a simplex spanned by a (proper) subset of $\{a_0, \dots, a_n\}$.*

Example 1.2.

(1) A 1-simplex, $a_0 a_1$, is a line segment, a 2-simplex, $a_0 a_1 a_2$, is a triangle, a 3-simplex, $a_0 a_1 a_2 a_3$ is a tetrahedron, etc.
(2) The points a_0, a_1, a_2 are geometrically independent if they are distinct and not collinear.
(3) Midpoint of $a_0 a_1$ has barycentric coordinates $(1/2, 1/2)$.
(4) Let $a_0 \dots a_3$ be a 3-simplex, then the proper faces are the simplexes $a_{i_1} a_{i_2} a_{i_3}, a_{i_4} a_{i_5}, a_{i_6}$ where $0 \leqslant i_1, \dots, i_6 \leqslant 3$.

Definition 1.3. *Two topological spaces X and Y are* ***homeomorphic*** *if there is a continuous bijection $f : X \longrightarrow Y$ with continuous inverse.*

We will glue simplexes together to get a **simplicial complex**. Fix N and $\mathbb{R}^N$.

Definition 1.4. *A (finite)* ***simplicial complex*** *K is a finite collection of simplexes in $\mathbb{R}^N$ satisfying the following axioms:*

(1) If σ is a simplex in K, then so is every face.
(2) If σ, τ are simplexes in K, then $\sigma \cap \tau$ is a face of both σ and τ.

Example 1.5. Octahedral surface: $8 \times$ 2-simplexes $\cup 12 \times$ 1-simplexes $\cup 6 \times$ 0-simplexes. One can check that this satisfies the above axioms.

Proposition 1.6. *Let K be a (finite) simplicial complex. A* ***sub-complex*** *J of K is a* ***subcollection*** *of simplexes in K such that if σ is a simplex in J, then so is every face of σ. In this case, axiom 2 holds for J automatically, so J is a simplicial complex.*

For $p \in \mathbb{N}$, then the p **-skeleton** of K is the subcomplex $K^{(p)}$ consisting of simplexes in K of dimension $\leqslant p$. The 0-skeleton $K^{(0)}$ is called the set of **vertices** of K. Note that K is determined by $K^{(0)}$ and collection of vertices which span simplexes of K.

Definition 1.7. *The* ***polytope*** *$|K|$ of a simplicial complex K in $\mathbb{R}^N$ is the union of the simplexes in K with topology induced from $\mathbb{R}^N$. Note that if K is an infinite dimensional simplex, then the usual topology from $\mathbb{R}^N$ is the wrong one to put on K.*

Example 1.8. Let $\sigma = a_0 \dots a_n$ be an n-simplex, then K is σ together with all its faces. Here $|K|$ is homeomorphic to a unit ball

$$\begin{aligned} B^n &= \{x \in \mathbb{R}^n \mid |x| \leqslant 1\} \\ |K^{(n-1)}| &= \text{boundary of } \sigma \text{ homeomorphic to } S^{n-1} = \{x \in \mathbb{R}^n \mid |x| = 1\} \end{aligned}$$

2. Simplicial maps and quotients

Lemma 2.1. *Let K, L be simplicial complexes and $f : K^{(0)} \longrightarrow L^{(0)}$ be a set map such that if $a_0, \dots, a_n$ span a simplex of K, then $f(a_0), \dots, f(a_n)$ span a simplex of L. Then there is a continuous function $g : |K| \longrightarrow |L|$. We call the function f its extension constructed below g a* ***simplicial map****.*

Proof. We have a map

$$\begin{array}{rcl} \beta_{a_0 \dots a_n} : \mathbb{R}^{n+1} & \longrightarrow & \mathbb{R}^N \ni a_0, \dots, a_n \\ (\lambda_0, \dots, \lambda_n) & \longmapsto & \lambda_0 a_0 + \dots + \lambda_n a_n \end{array}$$

which induces a homeomorphism from $e_1 \dots e_{n+1} \longrightarrow a_0 \dots a_n$ if $a_0, \dots, a_n$ are geometrically independent. Define g on each simplex $a_0 \dots a_n$ of $|K|$ by the composition

$$g : a_0 \dots a_n \stackrel{\beta^{-1}_{a_0 \dots a_n}}{\longrightarrow} e_1 \dots e_{n+1} \stackrel{\beta_{f(a_0) \dots f(a_n)}}{\longrightarrow} f(a_0) \dots f(a_n) \subseteq |L|$$

□

Corollary 2.2. *Let K and L be simplicial complexes and suppose there is a bijection $f : K^{(0)} \longrightarrow L^{(0)}$ such that $a_0, \dots, a_n$ span a simplex of K iff $f(a_0), \dots, f(a_n)$ span a simplex of L. Then the simplicial map $g : |K| \longrightarrow |L|$ above is an homeomorphism.*

The opposite procedure to constructing a polytope is triangulation. A **triangulation** of a topological space X is a simplicial complex K and a homeomorphism $g : |K| \longrightarrow X$. How do we triangulate topological spaces?

Proposition 2.3. *Let X be a topological space and Y be a set, and suppose π is a surjective map $X \longrightarrow Y$. The* ***quotient topology*** *on Y is defined by*

- *$U \subseteq Y$ is open iff $\pi^{-1}(U)$ is open in X.*

This is the weakest topology on Y which makes π continuous.

Remark 2.4. Often Y will be defined from X by putting an equivalence relation on the points of X.

Example 2.5. Torus $\mathbb{T}^2$. Let X be a rectangle in $\mathbb{R}^2$ with interior and $Y = X/\sim$ where $x \sim x'$ iff x and x' are opposite points on opposite sides of X. The quotient topology induced by the surjective map $X \longrightarrow Y$ makes Y homeomorphic to the 2-dimensional torus $\mathbb{T}^2$. A triangulation of Y is given by drawing vertices, edges and triangle on X, but care must be taken to ensure the choice of vertices etc result in a simplicial complex on Y.

3. Homology groups

Definition 3.1. *The* ***free abelian group*** *F with basis $\{s_i\}_{i \in I}$ is the direct sum*

$$F = \bigoplus_{i \in I} \mathbb{Z} s_i$$

Given an abelian group A and $a_i \in A$ for each $i \in I$, there is a group morphism

$$\begin{array}{rcl} \varphi : F & \longrightarrow & A \\ s_i & \longmapsto & a_i \end{array}$$

extended to all of F via $\mathbb{Z}$-linearity.

3.1. **Orientation.** An n-simplex $a_0 \dots a_n$ is independent of the order of vertices. Two orderings of $a_0, \dots, a_n$ are **equivalent** if they differ by an even permutation. An equivalence of orderings is called an **orientation**.

Definition 3.2. *An* ***oriented n -simplex*** *is an n-simplex $a_0 \dots a_n$ together with a choice of orientation of its vertices. If the orientation is $a_0 < \dots < a_n$, then we denote the oriented simplex by $[a_0 a_1 \dots a_n]$. Also, write $[a_1 a_0 a_2 \dots a_n] = -[a_0 a_1 \dots a_n]$.*

Example 3.3. For $n = 1$, $[a_0 a_1] \neq [a_1 a_0]$. For $n = 2$, $[a_0 a_1 a_2] = [a_1 a_2 a_0] = [a_2 a_0 a_1]$ and $[a_1 a_0 a_2] = [a_0 a_2 a_1] = [a_2 a_1 a_0]$.

3.2. **Group of (oriented) p-chains.** Let K be a simplicial complex. Pick a total order $<$ on the vertex set $K^{(0)}$. This induces an orientation of every simplex of K.

Definition 3.4. *The* ***group of oriented p -chains****, denoted $C_p(K)$, is the free abelian group with basis as the oriented p-simplexes of K with orientation induced by $<$. Define $C_{-1}(K) := 0$.*

Note 3.5.

(1) Suppose $[a_0 \dots a_p]$ is a simplex in K oriented by $<$, then

$$[a_1 a_0 a_2 \dots a_p] \quad = \quad -[a_0 \dots a_p] \in C_p(K)$$

so is an oriented p-chain.

(2) The group $C_p(K)$ is independent of the choice of the ordering $<$, changing $<$ will only change some basis elements to their negatives.

3.3. **Boundary operator.**

Proposition 3.6. *There is a group homomorphism*

$$\partial_p : C_p(K) \quad \longrightarrow \quad C_{p-1}(K)$$

which is uniquely defined by

$$\partial_p([a_0 \dots a_p]) \quad = \quad \sum_{i=1}^{p} (-1)^i [a_0 \dots \widehat{a_i} \dots a_p] \in C_{p-1}(K) \tag{1}$$

Proof. Since $C_p(K)$ is a free abelian group, we only need to specify ∂_p on the its basis, which is done by (1). It suffices to show that the right hand side of (1)

(1) remains unchanged if $a_0, \dots, a_p$ is permuted by an even permutation
(2) changes sign if $a_0, \dots, a_p$ is permuted by an odd permutation.

Note that S_{p+1} is generated by transpositions of the form $(j, j+1)$, so it suffices to show

$$\partial_p([a_0 \dots a_j a_{j+1} \dots a_p]) \quad = \quad -\partial_p([a_0 \dots a_{j-1} a_{j+1} a_j a_{j+2} \dots a_p]) \tag{2}$$

We consider the i-th summand in the right hand side of (1). When $i \neq j$ or $j+1$, both sides of (2) has the same i-th summand. Now consider the sum of the j-th and $(j+1)$-th summand of (2). We have,

$$\begin{aligned} j\text{-th} + (j+1)\text{-th terms of } l.h.s. \text{ of } (2) &= (-1)^j([a_0 \dots \hat{a}_j a_{j+1} \dots a_p] - [a_0 \dots a_j \hat{a}_{j+1} \dots a_p]) \\ j\text{-th} + (j+1)\text{-th terms of } r.h.s. \text{ of } (2) &= -(-1)^j([a_0 \dots a_j \hat{a}_{j+1} a_j a_{j+2}] - [a_0 \dots a_j a_{j+1} \hat{a}_j a_{j+2}]) \end{aligned}$$

and these are the same. □

Example 3.7. Consider a 2-simplex $[a_0 a_1 a_2]$, then

$$\partial_2([a_0 a_1 a_2]) \quad = \quad [a_1 a_2] - [a_0 a_2] + [a_0 a_1]$$

which is the oriented boundary of the triangle $a_0 a_1 a_2$. Also

$$\begin{aligned} \partial^2([a_0 a_1 a_2]) &= \partial([a_1 a_2] - [a_0 a_2] + [a_0 a_1]) \\ &= -a_2 + a_1 - (a_2 - a_0) + (a_1 - a_0) \\ &= 0 \end{aligned}$$

The formula $\partial^2 = 0$ holds in general for all p-chains.

$$\begin{aligned} \partial\left(\partial[a_0 \dots a_p]\right) &= \partial\left(\sum_{i=0}^{p} (-1)^i [a_0 \dots \widehat{a_i} \dots a_p]\right) \\ &= \sum_{i=0}^{p} (-1)^i \left(\sum_{j<i} (-1)^j [a_0 \dots \widehat{a_j} \dots \widehat{a_i} \dots a_p] + \sum_{j>i} (-1)^{j+1} [a_0 \dots \widehat{a_i} \dots \widehat{a_j} \dots a_p]\right) \\ &= 0 \end{aligned}$$

This result turns out to have important consequences.

4. Homology groups

Let K be a simplicial complex. The **group of p-cycles** of K is

$$Z_p(K) \quad = \quad \ker(\partial_p : C_p(K) \longrightarrow C_{p-1}(K)) \subset C_p(K)$$

The **group of p-boundaries** of K is

$$B_p(K) \quad = \quad \operatorname{im}(\partial_{p+1} : C_{p+1}(K) \longrightarrow C_p(K) \subset C_p(K).$$

The result $\partial^2 = 0$ implies

$$\partial_p B_p(K) \quad = \quad \partial_p \partial_{p+1} C_{p+1}(K)$$

hence $B_p(K) \subseteq Z_p(K)$. So we can define the p **-th homology group** of K to be the quotient

$$H_p(K) \quad = \quad Z_p(K)/B_p(K).$$

4.1. **Numerical invariants.** Note that $C_p(K)$ is finitely generated abelian, hence $Z_p(K)$ and $B_p(K)$ are both finitely generated abelian. So

$$H_p(K) \quad = \quad \mathbb{Z}^s \times \mathbb{Z}/h_1\mathbb{Z} \times \ldots \times \mathbb{Z}/h_r\mathbb{Z}$$

and we say that the integer s is the p **-th Betti number** of K, denoted $b_p(K)$. The group $H_p(K)$ (actually $b_p(K)$) measures the "holes" in $|K|$.

Example 4.1. Let K be a one dimensional triangle, with vertices a, b, c. We have

$$\begin{array}{ccccccc} C_2(K) & \xrightarrow{\partial_2} & C_1(K) & \xrightarrow{\partial_1} & C_0(K) & \xrightarrow{\partial_0} & 0 \\ || & & \downarrow & & \downarrow & & \\ 0 & & \mathbb{Z}\,[ab] \oplus \mathbb{Z}\,[bc] \oplus \mathbb{Z}\,[ca] & & \mathbb{Z}a \oplus \mathbb{Z}b \oplus \mathbb{Z}c & & \end{array}$$

where the vertical maps are isomorphisms. The map ∂_1 is defined on the basis by

$$\begin{array}{rcl} [ab] & \longmapsto & b - a \\ [bc] & \longmapsto & c - b \\ [ca] & \longmapsto & a - c \end{array}$$

There are two nonzero homology groups,

$$\begin{array}{rcl} H_0(K) & \simeq & C_0(K)/\operatorname{im}(\partial_1) \\ & \simeq & \dfrac{\mathbb{Z}a \oplus \mathbb{Z}b \oplus \mathbb{Z}c}{(b-a, c-b, a-c)} \\ & \simeq & \mathbb{Z} \\ H_1(K) & \simeq & \ker(\partial_1) \\ & \simeq & \mathbb{Z}([ab] + [bc] + [ca]) \\ & \simeq & \mathbb{Z} \end{array}$$

Proposition 4.2. *Let K be a simplicial complex such that $|K|$ has n connected components. Then $H_0(K) \simeq \mathbb{Z}^n$.*

Proof. Generalise calculation of $H_0(K)$ above. □

Definition 4.3. *Let K be a simplicial complex. We say that two p-chains γ, γ' are **homologous** if $\gamma - \gamma'$ is a p-boundary. Let L be a subcomplex of K, note that $C_p(L) \subset C_p(K)$, we say $\gamma \in C_p(K)$ is **carried** by L if $\gamma \in C_p(L)$.*

Example 4.4. Let $|T|$ be the two torus $\mathbb{T}^2$. Let T be the complex below

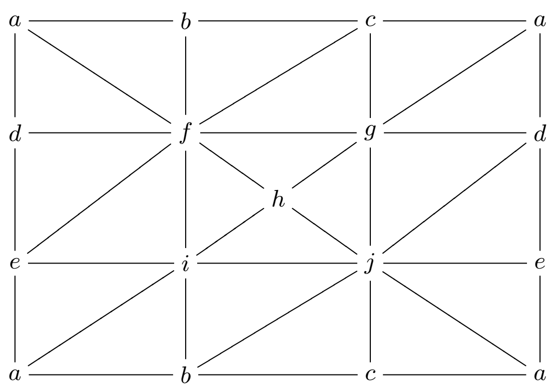


Figure 1: A triangulation of the torus.

where the opposite sides are identified. Let L be the "boundary" of the above diagram. Let $\{\sigma_r\}$ be the set of 2-simplexes in T, oriented anticlockwise. Let γ be a 2-chain such that $\partial\gamma$ is carried by L. Then

(1) $\gamma = n\sum\sigma_r$ for some $n \in \mathbb{Z}$.
(2) $H_2(T) = Z_2(T) = \mathbb{Z}\sum\sigma_r \simeq \mathbb{Z}$.

Let $\gamma = \sum n_r\sigma_r$, Suppose σ_r and σ_s are adjacent with common "internal (not in L)" edge ε. The coefficient of ε in $\partial\gamma$ is $\pm(n_r - n_s)$. Hence $\partial\gamma$ carried by L implies $n_r = n_s$, so by induction all n_r are the same. This proves part 1. Note that $\partial\sum\sigma_r = 0$ hence $\sum\sigma_r \in Z_2(T)$. So part 1 implies $Z_2(T) = \mathbb{Z}\sum\sigma_r = H_2(T)$. This proves part 2.

Proposition 4.5. *Let $\alpha = [ab] + [bc] + [ca]$, $\beta = [ad] + [de] + [ea]$*

(1) Any 1-cycle in T is homologous to one carried by L.
(2) $Z_1(L) = C_1(L) \cap Z_1(T) = \mathbb{Z}\alpha + \mathbb{Z}\beta$, and $H_1(T) \simeq \mathbb{Z}^2$.

Proof. Let $\gamma \in Z_1(T)$. By adding appropriate multiples of $\partial\sigma_r$'s, where $\sigma_r = [fih]$, we can adjust γ so that the coefficient of $[ih]$ is zero. Continuing inductively to find a homologous cocycle which is carried by

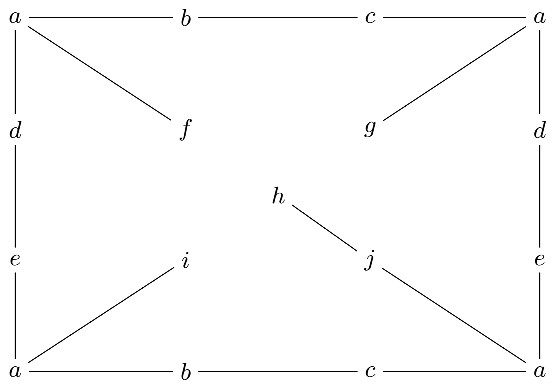


Figure 2: After deletion of relevant 1-boundaries.

But γ' is a 1-cycle, so in fact, it is carried by L. For part 2,

$$\begin{aligned} H_1(T) &= Z_1(T)/B_1(T) \\ \text{part 1} &= \frac{Z_1(L) + B_1(T)}{B_1(T)} \\ \text{example 4.4, 1} &= \frac{Z_1(L)}{Z_1(L) \cap B_1(T)} \\ &= 0 \end{aligned}$$

As in example 1 we can show that $Z_1(L) = \mathbb{Z}\alpha + \mathbb{Z}\beta = \mathbb{Z}\alpha \oplus \mathbb{Z}\beta$. □

5. Chain complexes

Definition 5.1. *A **chain complex** of abelian groups is a sequence*

$$C_\bullet : \ldots \longrightarrow C_{p+1} \xrightarrow{\partial_{p+1}} C_p \xrightarrow{\partial_p} C_{p-1} \longrightarrow \ldots$$

of abelian groups C_p, $p \in \mathbb{Z}$ and group homomorphisms ∂_p such that $\partial_p \circ \partial_{p+1} = 0$ for all p.

Remark 5.2. Later we will talk of chain complexes of vector spaces, (C_p are vector spaces and ∂_p are linear maps), or of modules (so C_p are modules and ∂_p are module morphisms).

Definition 5.3. *Let $C_\bullet$ be a chain complex. The **group of** p **-cycles** is $Z_p(C_\bullet) := \ker(\partial_p) \subseteq C_p$ and the **group of** p **-boundaries** is $B_p(C_\bullet) := \mathrm{im}(\partial_{p+1}) \subseteq C_p$. Since $\partial_p \circ \partial_{p+1} = 0$ so $B_p(C_\bullet) \subseteq Z_p(C_\bullet)$ so we can define the p **-th homology group***

$$H_p(C_\bullet) = Z_p(C_\bullet)/B_p(C_\bullet).$$

Example 5.4.

(1) Let K be a simplicial complex and $C_\bullet = C_\bullet(K)$ is a chain complex.
(2) Augmented chain complex of K. Define

$$\tilde{C}_\bullet(K) = \begin{cases} C_p(K) & p \neq -1 \\ \mathbb{Z} & p = -1 \end{cases}$$

Also ∂_p is the same as for $C_\bullet(K)$ when $p \neq 0, -1$. For $p = -1$ we define $\partial_{-1} : \tilde{C}_{-1} \longrightarrow \tilde{C}_{-2}$ to be the zero map and $\partial_0 : \tilde{C}_0 \longrightarrow \tilde{C}_{-1}$ by $[a] \longmapsto 1$ for all $[a] \in \tilde{C}_0$. This is also a chain complex, since

$$\partial_1 \partial_0 [ab] \quad = \quad \partial_0(b - a) \; = \; 0.$$

This allows us to define the ***p* -th reduced homology groups**, $\tilde{H}_p(K) := H_p(\tilde{C}_\bullet)$ of K. Note that $H_p(K) = \tilde{H}_p(K)$ for all $p \neq 0$.

Definition 5.5. *A **cochain complex** of abelian groups is a sequence*

$$C^\bullet : \ldots \longrightarrow C^{p-1} \xrightarrow{d^{p-1}} C^p \xrightarrow{d^p} C^{p+1} \longrightarrow \ldots$$

of abelian groups C^p, $p \in \mathbb{Z}$ and group homomorphisms ∂^p such that $\partial^p \circ \partial^{p-1} = 0$ for all p.

Definition 5.6. *Let $C^\bullet$ be a cochain complex. The **group of** p **-cocycles** is $Z^p(C^\bullet) := \ker(d^p) \subseteq C^p$ and the **group of** p **-coboundaries** is $B^p(C^\bullet) := \mathrm{im}(d^{p-1}) \subseteq C^p$. Since $\partial^p \circ \partial^{p-1} = 0$ so $B^p(C^\bullet) \subseteq Z^p(C^\bullet)$ so we can define the p **-th cohomology group***

$$H^p(C^\bullet) \quad = \quad Z^p(C^\bullet)/B^p(C^\bullet).$$

Example 5.7. The obstruction to some curl free vector field F being conservative is in the first de Rham cohomology group

$$H^1_{\mathrm{DR}}(X) \quad = \quad \frac{\ker(\mathrm{curl})}{\mathrm{im}(\mathrm{grad})}.$$

Definition 5.8. *Let $C_\bullet$ and $C'_\bullet$ are two chain complexes. A **chain map** or **morphism** of chain complexes is a sequence of maps $\{f_p\}_{p \in \mathbb{Z}}$ where $f_p \in \mathrm{Hom}_{\mathbb{Z}}(C_p, C'_p)$ for each p and that the diagram*

$$\begin{array}{ccccccccc}
\ldots & \longrightarrow & C_{p+1} & \xrightarrow{\partial_{p+1}} & C_p & \xrightarrow{\partial_p} & C_{p-1} & \longrightarrow & \ldots \\
 & & \downarrow f_{p+1} & & \downarrow f_p & & \downarrow f_{p-1} & & \\
\ldots & \longrightarrow & C'_{p+1} & \xrightarrow{\partial'_{p+1}} & C'_p & \xrightarrow{\partial'_p} & C'_{p-1} & \longrightarrow & \ldots
\end{array}$$

commutes. This is denoted

$$f_\bullet : C_\bullet \quad \longrightarrow \quad C'_\bullet$$

5.1. **Functoriality.**

Proposition 5.9. *Let $f_\bullet : C_\bullet \longrightarrow C'_\bullet$ be a chain map. Then*

(1) we have inclusions $f_p(Z_p(C_\bullet)) \subseteq Z_p(C'_\bullet)$ and $f_p(B_p(C_\bullet)) \subseteq B_p(C'_\bullet)$, and
(2) $f_\bullet$ induces a map on homology,

$$\begin{array}{rcl}
f_* = H_p(f_\bullet) : H_p(C_\bullet) & \longrightarrow & H_p(C'_\bullet) \\
{[\varphi]} & \longmapsto & [f_p \varphi]
\end{array}$$

(3) $\mathrm{id}_* = \mathrm{id}$
(4) Given another chain map $g_\bullet : C'_\bullet \longrightarrow C''_\bullet$, then the collection $\{g_p \circ f_p\}$ is a chain map

$$g_\bullet f_\bullet : C'_\bullet \quad \longrightarrow \quad C''_\bullet$$

and the following diagram

$$\begin{array}{ccccc}
H_p(C_\bullet) & & \xrightarrow{(gf)_*} & & H_p(C''_\bullet) \\
 & \searrow^{f_*} & & \nearrow^{g_*} & \\
 & & H_p(C'_\bullet) & &
\end{array}$$

commutes.

Proof. Let $\gamma \in Z_p(C_\bullet)$ then

$$\partial' f(\gamma) \; = \; f \partial(\gamma) \quad = \quad f(0) \; = \; 0$$

so $f(\gamma) \in Z_p(C'_\bullet)$. Also

$$f(B_p(C_\bullet)) \; = \; f \partial C_{p+1} \quad = \quad \partial' f C_{p+1} \subseteq B_p(C').$$

□

6. Homotopy and homology of cones and sphers

Definition 6.1. *Let $f_\bullet$ and $g_\bullet$ be two chain maps $C_\bullet \longrightarrow C'_\bullet$, a **chain homotopy** between $f_\bullet$ and $g_\bullet$ is a collection of degree 1 morphisms $s_p : C_p \longrightarrow C'_{p+1}$ satisfying*

$$f_p - g_p \quad = \quad \partial'_{p+1} s_p + s_{p-1}\partial_p.$$

*In this case we say that $f_\bullet$ and $g_\bullet$ are **chain homotopic**.*

Proposition 6.2. *Chain homotopic maps $f_\bullet, g_\bullet : C_\bullet \longrightarrow C'_\bullet$ induce the same map in homology. In particular, if $f_\bullet$ is homotopic to the identity, then $f_\bullet$ is a quasi-isomorphism[1]. Also, if the identity is homotopic to zero, then $C_\bullet$ is an exact[2] complex.*

Proof. Let $\{s_p\}$ be the chain homotopy between $f_\bullet$ and $g_\bullet$, then

$$\begin{aligned} f_*([\gamma]) &= [f\gamma] \\ &= [\partial'_{p+1} s_p\gamma + s_{p-1}\partial_p.\gamma + g\gamma] \\ \text{since } \gamma \text{ is a cycle} &= [\partial'_{p+1} s_p\gamma + g\gamma] \\ \text{since } \partial'_{p+1} s_p\gamma \text{ is a boundary} &= [g\gamma] \end{aligned}$$

hence $f_* = g_*$. □

6.1. Cones.

Let K be a simplicial complex in $\mathbb{R}^N$. Let $w \in \mathbb{R}^N$ such that if $a_0, \dots, a_p$ is a simplex in K, then $w, a_0, \dots, a_p$ is geometrically independent. Such a w exists for sufficiently large N.

Definition 6.3. *The cone $w * K$ is the simplicial complex consisting of all simplexes of K and simplexes of the form $w\sigma$ for any simplex σ in K.*

Proposition 6.4. *With notation as above,*
1. *$\tilde{H}_p(w * K) = 0$*
2. *$H_p(w * K) = 0$ for $p > 0$ and $H_0(w * K) = \mathbb{Z}$.*

Proof. Since $w * K$ is connected, $H_0(w * K) = \mathbb{Z}$ by proposition 4.2. Also by example 5.4, 2, $\tilde{H}_p(w * K) = H_p(w * K)$ if $p \neq 0$ so it suffcies to prove part 1. By proposition 6.2, it suffices to show id and 0 are chain homotopic. Define a chain homotopy by

$$\begin{aligned} s_p : \tilde{C}_p(w * K) &\longrightarrow \tilde{C}_{p+1}(w * K) \\ \sigma &\longmapsto [w\sigma] \\ w\sigma &\longmapsto 0 \end{aligned}$$

where σ is an oriented simplex in K. Note that $\partial[w\sigma] = \sigma - [w\partial\sigma]$, then

$$\begin{aligned} (s\partial + \partial s)\sigma &= [w\partial\sigma] + \partial[w\sigma] \\ &= [w\partial\sigma] + \sigma - [w\partial\sigma] \\ &= \sigma \end{aligned}$$

$$\begin{aligned} (s\partial + \partial s)[w\sigma] &= s\sigma - s[w\partial\sigma] \\ &= [w\sigma] \end{aligned}$$

hence $s\partial + \partial s = \text{id}$ giving proof of part 1. □

Theorem 6.5. *Let σ be an n-simplex.*
1. *Let K_σ be the simplicial complex consisting of all faces of σ. Then $\tilde{H}_p(K_\sigma) = 0$.*
2. *Let $\Sigma^{n-1} = K_\sigma^{(n-1)}$, then*

$$H_p(\Sigma^{n-1}) \quad = \quad \begin{cases} \mathbb{Z} & p = 0 \text{ or } n-1 \\ 0 & \text{otherwise} \end{cases}$$

Proof. The proof of part 1 is an exercise using induction on n and proposition 6.4. For part 2, $H_0(\Sigma^{n-1}) = \mathbb{Z}$ since $|\Sigma^{n-1}|$ has one connected component. Consider a chain map

$$\begin{array}{ccccccccc} & & 0 & \longrightarrow & C_{n-1}(\Sigma^{n-1}) & \longrightarrow & C_{n-2}(\Sigma^{n-1}) & \longrightarrow & \dots \\ & & \downarrow & & || & & || & & \\ \mathbb{Z}\sigma & = & C_n(K_\sigma) & \longrightarrow & C_{n-1}(K_\sigma) & \longrightarrow & C_{n-2}(K_\sigma) & \longrightarrow & \dots \end{array}$$

We see that $H_p(\Sigma^{n-1}) = 0$ for $p \neq n-1, 0$. It remains to check $p = n-1$,

$$H_{n-1}(\Sigma^{n-1}) \;=\; Z_{n-1}(\Sigma^{n-1}) \;=\; Z_{n-1}(K_\sigma) \;=\; B_{n-1}(K_\sigma) \;=\; \mathbb{Z}\partial\sigma$$

□

---

[1] $H_p(f_\bullet)$ is an isomorphism for all p.
[2] All cohomology vanish.

7. Relative homology

Let K be a simplicial complex and L be a subcomplex of K. Note that $C_p(L)$ is a subgroup of $C_p(K)$. This gives an example of a subchain complex.

Definition 7.1. *Let $(C_\bullet, \partial)$ be a chain complex. A* **subchain complex** *$D_\bullet$ is a family of subgroups $D_p \subseteq C_p$ for all $p \in \mathbb{Z}$ such that $\partial(D_p) \subseteq D_{p-1}$. Note that $D_\bullet$ is naturally a chain complex and we have a chain map $D_\bullet \hookrightarrow C_\bullet$. We get an obvious induced* **quotient chain complex,** *$C_\bullet/D_\bullet$, and a chain map $C_\bullet \longrightarrow C_\bullet/D_\bullet$.*

Let L be a subcomplex of K. The group of **relative p-chains** of K modulo L is defined as

$$C_p(K, L) \quad = \quad C_p(K)/C_p(L)$$

and the **relative p-th homology** is

$$H_p(K, L) \quad = \quad H_p(C_\bullet(K)/C_\bullet(L))$$

What is the topological significance of this? Let X be the quotient space of $|K|$ where we identify all points of $|L|$ together. It turns out that usually $H_p(K, L)$ is the "reduced homology of X."

Example 7.2. Let K =all faces of an n-simplex σ, $L = K^{(n-1)}$, so $|L|$ is homeomorphic to S^{n-1}, and $X = n$-sphere.

$$\begin{array}{lcccccccccc}
C_\bullet(L) & : & 0 & \longrightarrow & 0 & \longrightarrow & C_{n-1}(\Sigma^{n-1}) & \longrightarrow & C_{n-2}(\Sigma^{n-1}) & \longrightarrow & \dots \\
 & & & & \downarrow & & || & & || & & \\
C_\bullet(K) & : & 0 & \longrightarrow & \mathbb{Z}\sigma & \longrightarrow & C_{n-1}(K_\sigma) & \longrightarrow & C_{n-2}(K_\sigma) & \longrightarrow & \dots \\
 & & & & & & \downarrow & & \downarrow & & \\
C_\bullet(K, L) & : & 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \dots
\end{array}$$

So

$$H_p(K, L) \quad = \quad \begin{cases} \mathbb{Z} & p = n \\ 0 & \text{otherwise} \end{cases}$$

Exercise: this is the reduced homology of the n-sphere X.

7.1. The excision theorem.

Theorem 7.3. *Let K be a simplicial complex; L, K_0 are subcomplexes of K such that $U := |K| - |K_0|$ is contained in $|L|$; and L_0 be the subcomplex $L \cap K_0$. Then*

$$H_p(K, L) \quad \simeq \quad H_p(K_0, L_0).$$

Proof. Every simplex in K is a simplex in K_0 or L (or both), so $C_p(K) = C_p(K_0) + C_p(L)$, then

$$C_p(K, L) \; = \; \frac{C_p(K_0) + C_p(L)}{C_p(L)} \quad \simeq \quad \frac{C_p(K_0)}{C_p(K_0) \cap C_0(L)} \; \simeq \; \frac{C_p(K_0)}{C_p(L_0)} \; = \; C_p(K_0, L_0)$$

so the relative chain groups are the same and moreover the boundary operators coincide, hence we have the stated isomorphism. □

8. Simplicial maps and homology

Definition 8.1. *Let $f : K \longrightarrow L$ be a simplicial map. The* **induced chain map** *is the chain map defined by*

$$\begin{array}{rcl}
f_\# : C_\bullet(K) & \longrightarrow & C_\bullet(L) \\
{[a_0, \dots, a_p]} & \longmapsto & \begin{cases} [f(a_0), \dots, f(a_p)] & \textit{if } f(a_0), \dots, f(a_p) \textit{ are distinct} \\ 0 & \textit{otherwise} \end{cases}
\end{array}$$

There is also an induced map

$$f_\# : \tilde{C}_\bullet(K) \quad \longrightarrow \quad \tilde{C}_\bullet(L)$$

with the same definition as above except

$$\begin{array}{rcl}
f_{\#,-1} : \tilde{C}_{-1}(K) & \longrightarrow & \tilde{C}_{-1}(L) \\
|| & & || \\
\mathbb{Z} & & \mathbb{Z}
\end{array}$$

is the identity.

We check that $f_\#$ commutes with ∂:

$$\begin{aligned} f_\#\partial([a_0,\ldots,a_p]) &= f_\#\sum_{i=0}^{p}(-1)^i\,[a_0,\ldots,\widehat{a_i},\ldots,a_p] \\ &= \sum_{i=0}^{p}(-1)^i\left[f(a_0),\ldots,\widehat{f(a_i)},\ldots,f(a_p)\right] \\ &= \partial f_\#([a_0,\ldots,a_p]) \end{aligned}$$

Exercise: check degenerate cases.

8.1. **Functoriality.**

Proposition 8.2. *Let $f : K \longrightarrow L$ be a simplicial map. Then*

(1) f induces a group morphism

$$f_* := H_p(f_\#) : H_p(K) \longrightarrow H_p(K)$$

and similarly for reduced homology.

(2) The identity map $K \longrightarrow K$ induces the identity in homology.

(3) Given another simplicial map $g : L \longrightarrow M$, the following diagram commutes

$$\begin{array}{ccc} (gf)_* : H_p(K) & \longrightarrow & H_p(M) \\ & {}_{f_*}\searrow\nearrow_{g_*} & \\ & H_p(L) & \end{array}$$

Proof. Follows from previous result, functoriality of chain maps (proposition 5.9), and the fact that $\mathrm{id}_\# = \mathrm{id}$ and $(gf)_\# = g_\# f_\#$. □

Example 8.3. Recall the diagram in example 4.4. Let $\tau = [x_0x_1x_2]$ be a 3-simplex consider the following maps from $\tau^{(1)}$ to T

$$\begin{aligned} \iota : \tau^{(1)} &\longrightarrow T \\ x_0 &\longmapsto a \\ x_1 &\longmapsto f \\ x_2 &\longmapsto i \end{aligned}$$

and

$$\begin{aligned} \iota : \tau^{(1)} &\longrightarrow T \\ x_0 &\longmapsto b \\ x_1 &\longmapsto f \\ x_2 &\longmapsto i; \end{aligned}$$

and the projection from T to $\ell^{(1)}$ where $\ell = [ade]$. Let $\alpha = [ab] + [bc] + [ca] + B_1(T)$, $\beta = [ad] + [de] + [ea] + B_1(T)$ be the generators of $H_1(T)$, then the homology maps are

$$\begin{array}{ccccc} \mathbb{Z} & \xrightarrow{\iota_*, i_*} & \mathbb{Z}\alpha \oplus \mathbb{Z}\beta & \xrightarrow{\pi_*} & \mathbb{Z} \\ 1 & \longmapsto & \beta \text{ (or } -\beta) & & \\ & & \beta & \longmapsto & 1 \\ & & \alpha & \longmapsto & 0 \end{array}$$

Definition 8.4. *Two simplicial maps $f, g : K \longrightarrow L$ are **contiguous** if for every simplex $a_0, \ldots, a_p \in K$, $f(a_0), \ldots, f(a_p), g(a_0), \ldots, g(a_p)$ span a simplex of L.*

Example 8.5. In the above example, i and ι are continguous.

Definition 8.6.

*(1) A chain complex $C_\bullet$ is **acyclic** if all its homology vanishes. A simplicial complex K is **acyclic** if $\tilde{C}_\bullet(K)$ is.*

*(2) Let L be another simplicial complex. An **acyclic carrier** Φ from K to L is a family $\{\Phi(\sigma)\}_{\sigma\in K}$ of subcomplexes of L such that*

(a) $\Phi(\sigma)$ is nonempty acyclic.

(b) If τ is a face of σ then $\Phi(\tau)$ is a subcomplex of $\Phi(\sigma)$.

*(3) A homomorphism $f : C_p(K) \longrightarrow C_p(K)$ is **carried by** Φ if for every simplex $\sigma \in K$, $f(\sigma)$ is carried by $\Phi(\sigma)$.*

Example 8.7. Any cone $w * K$ is acyclic.

Proposition 8.8. *Let $f, g : K \longrightarrow L$ be continguous simplicial maps. Then there is a chain homotopy between $f_\#$ and $g_\# : C_\bullet(K) \longrightarrow C_\bullet(L)$. Similarly, for $f_\#$ and $g_\# : \tilde{C}_\bullet(K) \longrightarrow \tilde{C}_\bullet(L)$.*

Proof. We can consider the acyclic carrier Γ from K to L defined by

$$\Phi([a_0, \dots, a_p]) \quad = \quad f(a_0) \dots f(a_p) g(a_0) \dots g(a_p)$$

that is, all faces of this simplex in L. Note that $f_\#$ and $g_\#$ are carried by Φ. This proposition follows from the following *acyclic carrier theorem.* □

Theorem 8.9. *Let Φ be an acyclic carrier from K to L. Let $\tilde{f}, \tilde{g} : \tilde{C}_\bullet(K) \longrightarrow \tilde{C}_\bullet(L)$ be chain maps carried by Φ, such that $\tilde{f}_{-1} = \tilde{g}_{-1}$. Let $f, g : C_\bullet(K) \longrightarrow C_\bullet(L)$, be the restriction of $\tilde{f}$ and $\tilde{g}$. Then there is a chain homotopy s, carried by Φ, between $\tilde{f}$ and $\tilde{g}$ (and also between f and g).*

Proof. We do the $\tilde{f}, \tilde{g}$ case first. We construct the homotopy $s_p : \tilde{C}_p(K) \longrightarrow \tilde{C}_{p+1}(L)$ inductively on p. For the case $p = -1$: we have the following diagram

$$\begin{array}{ccccccc} \text{degree} & & 0 & & -1 & & -2 \\ \dots & \longrightarrow & C_0(K) & \xrightarrow{\partial} & \mathbb{Z} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \\ \dots & \longrightarrow & C_0(L) & \xrightarrow{\partial} & \mathbb{Z} & \longrightarrow & 0 \end{array}$$

since $s_{-2} = 0$, homotopy means $\tilde{f} - \tilde{g} = s\partial + \partial s$, $\tilde{f}_{-1} - \tilde{g}_{-1} = 0$ implies $s_{-1} = 0$ will do. For $p \geqslant 0$, we need

$$\tilde{f}([a_0 \dots a_p]) - \tilde{g}([a_0 \dots a_p]) \quad = \quad \partial_{p+1} s_p([a_0 \dots a_p]) + s_{p-1}\partial_p([a_0 \dots a_p])$$

so it suffices to solve for $s_p([a_0 \dots a_p]) = z$ a chain in $\Phi([a_0 \dots a_p])$:

$$\partial z \quad = \quad -s_{p-1}\partial([a_0 \dots a_p]) + \tilde{f}([a_0 \dots a_p]) - \tilde{g}([a_0 \dots a_p]). \tag{3}$$

Now $\tilde{f}, \tilde{g}$ and s_{p-1} are carried by Φ and $\Phi([a_0 \dots a_p])$ is acyclic. It suffices to show that the *r.h.s.* of (3) is acyclic. By induction,

$$\begin{aligned} \partial s_{p-1}\partial([a_0 \dots a_p]) &= (s\partial + \tilde{f} - \tilde{g})\partial([a_0 \dots a_p]) \\ &= (\tilde{f} - \tilde{g})\partial([a_0 \dots a_p]) \\ &= \partial\tilde{f}([a_0 \dots a_p]) - \partial\tilde{g}([a_0 \dots a_p]) \end{aligned}$$

so ∂ of the *r.h.s.* of (3) is zero, and we can solve (3) to get case $\tilde{f}, \tilde{g}$. To get the case f, g note that as $s_{-1} = 0$, we can just restrict s in $\tilde{f}, \tilde{g}$ case. □

9. Categories

Definition 9.1. *A **category** $\mathcal{C}$ consists of the data*

*(1) A class of **objects** $\mathrm{Ob}(\mathcal{C})$.*
*(2) (Disjoint) sets of **morphisms** $\mathrm{Hom}_\mathcal{C}(X, Y)$ for any $X, Y \in \mathrm{Ob}(\mathcal{C})$,*

satisfying

a) For any $X, Y, Z \in \mathrm{Ob}(\mathcal{C})$, the composition map

$$\begin{array}{rcl} \mathrm{Hom}_\mathcal{C}(X, Y) \times \mathrm{Hom}_\mathcal{C}(Y, Z) & \longrightarrow & \mathrm{Hom}_\mathcal{C}(X, Z) \\ (f, g) & \longmapsto & g \circ f \end{array}$$

is associative.

b) For each $X \in \mathrm{Ob}(\mathcal{C})$ there is a morphism $\mathrm{id}_X \in \mathrm{Hom}_\mathcal{C}(X, X)$ such that $\mathrm{id}_X \circ g = g$, $f \circ \mathrm{id}_X = f$ when defined.

Example 9.2.

(1) The category $\mathfrak{Set}$ consists of $\mathrm{Ob}(\mathfrak{Set})$ =class of sets, $\mathrm{Hom}_{\mathfrak{Set}}(X, Y)$ =set of functions from X to Y, with composition being the usual function composition.
(2) The category $\mathfrak{Grp}$ consists of $\mathrm{Ob}(\mathfrak{Grp})$ =class of groups, $\mathrm{Hom}_{\mathfrak{Grp}}(X, Y)$ =group morphisms.
(3) The category $\mathfrak{Top}$ consists of $\mathrm{Ob}(\mathfrak{Top})$ =class of topological spaces, $\mathrm{Hom}_{\mathfrak{Top}}(X, Y)$ =continuous maps.
(4) (Nonstandard notation to follow.) The category $\mathfrak{SimpS}$ consists of $\mathrm{Ob}(\mathfrak{SimpS})$ =class of simplicial complexes, $\mathrm{Hom}_{\mathfrak{SimpS}}(X, Y)$ =simplicial maps.
(5) The category $\mathfrak{Ch}$ consists of $\mathrm{Ob}(\mathfrak{Ch})$ =class of chain complexes, $\mathrm{Hom}_{\mathfrak{Ch}}(C_\bullet, C'_\bullet)$ =chain maps.
(6) The category K consists of $\mathrm{Ob}(K)$ =class of chain complexes, $\mathrm{Hom}_K(C_\bullet, C'_\bullet) = \mathrm{Hom}_{\mathfrak{Ch}}(C_\bullet, C'_\bullet)/\sim$ where $f \sim g$ whenever f is homotopic to g. This is called the homotopic category.

Definition 9.3. *An **isomorphism** in a category $\mathcal{C}$ is a morpihsm $f \in \mathrm{Hom}_\mathcal{C}(X, Y)$ such that there there is $g \in \mathrm{Hom}_\mathcal{C}(Y, X)$ satisfying $g \circ f = \mathrm{id}_X$, $f \circ g = \mathrm{id}_Y$.*

Example 9.4. In the case $\mathcal{C} = K$, a chain map $f : C_\bullet \longrightarrow C'_\bullet$ is said to be a **chain equivalence** if the homotopy equivalence class $[f]$ containing it is an isomorphism in K. If the chain map $g : C'_\bullet \longrightarrow C_\bullet$ represents the inverse $[g]$ of $[f]$, then we say that f and g are chain-homotopy inverses. That is $f \circ g$ and $g \circ f$ are homotopic to the identity.

Definition 9.5. *A* ***covariant fucntor*** $F : \mathcal{C} \longrightarrow \mathcal{C}'$ *is the data consisting of*

(1) A function

$$\begin{array}{ccc} \mathrm{Ob}(\mathcal{C}) & \longrightarrow & \mathrm{Ob}(\mathcal{C}') \\ X & \longmapsto & F(X) \end{array}$$

(2) For any $X, Y \in \mathrm{Ob}(\mathcal{C})$

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{C}}(X,Y) & \longrightarrow & \mathrm{Hom}_{\mathcal{C}'}(F(X),F(Y)) \\ f & \longmapsto & F(f) \end{array}$$

satisfying

a) $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$
b) $F(g \circ f) = F(g) \circ F(f)$.

Example 9.6. The homology H_p is a functor from $\{\mathfrak{Ch}, \mathfrak{SimpS}, K\}$ to $\mathfrak{Grp}$.

Proposition 9.7. *Let* $F : \mathcal{C} \longrightarrow \mathcal{D}$ *be a covariant functor and* f *is an isomorphism in* $\mathcal{C}$. *Then* $F(f)$ *is an isomorphism in* $\mathcal{D}$.

Proof. Since $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$, let g be the left inverse of f then $\mathrm{id} = F(g \circ f) = F(g) \circ F(f)$. The case for right inverse is the same. □

Example 9.8. Suppose $f : C_\bullet \longrightarrow C'_\bullet$ is a chain equivalence, then the above proposition implies $H_p(f) = f_*$ is an isomorphism.

9.1. **Topological invariance.** The aim is to construct functors

$$H_p : \mathfrak{Top} \quad \longrightarrow \quad \mathfrak{Grp},$$

then we can distinguish topological spaces X and Y if we can show $H_p(X) \neq H_p(Y)$.

10. Subdivision

Definition 10.1. *Let* K *be a simplicial complex. A* ***subdivision*** K' *of* K *is a simplicial complex such that*

(1) every simplex of K' *is contained in a simplex of* K,
(2) every simplex σ *in* K *is a union of simplexes in* K'.

Let $K'(\sigma)$ *be the subcomplex of* K' *with* $|K'(\sigma)| = \sigma$.

10.1. **Barycentric subdivision.** Let $\sigma = a_0 \dots a_p$ be a p-simplex. The **barycentre** $\hat{\sigma}$ is the point in σ where all the barycentric coordinates are equal. (generalises midpoint and centroid). Let K be a simplicial complex. The **barycentric subdivision** of K is the subdivision $\mathrm{sd}\, K$ with vertices $\{\hat{\sigma} \mid \sigma \text{ a simplex in } K\}$, and simplexes of the form $\hat{\sigma}_1 \dots \hat{\sigma}_p$ where $\sigma_1 \supset \sigma_2 \supset \dots \supset \sigma_p$ are simplexes in K. Note that

$$(\mathrm{sd}\, K)(\sigma) \quad = \quad \hat{\sigma} * \mathrm{sd}\, K_\sigma^{(p-1)}$$

where $K_\sigma^{(p-1)}$ =boundar of the simplex σ, hence is acyclic. For a simplex σ, denote $\mathrm{Int}(\sigma)$ =interior of σ.

Lemma 10.2. *Let* K' *be a subdivision of the simplicial complex* K. *Let* $g : K'^{(0)} \longrightarrow K^{(0)}$ *be a function such that if* $v \in K'^{(0)}$ *is in* $\mathrm{Int}(\sigma)$ *where* $\sigma \in K$, *then* $g(v)$ *is a vertex of* σ. *Then* g *is a simplicial map.*

Remark 10.3. Such simplicial maps exist.

Proof. Let $\tau = a_0 \dots a_p$ be a p-simplex in K'. It is contained in some simplex σ of K, hence each a_o is in the interior of some face of σ. All $g(a_o)$ are some vertices of σ, so g is simplicial. □

10.2. **Subdivision theorem.**

Theorem 10.4. *Let* K' *be a subdivision of* K. *Let* $g : K' \longrightarrow K$ *be a simplicial map satisfying the hypothesis of lemma 10.2. There is a chain map*

$$\lambda : C_\bullet(K) \quad \longrightarrow \quad C_\bullet(K')$$

which is a chain homotopy inverse to $g_\# : C_\bullet(K') \longrightarrow C_\bullet(K)$. *In particular,* $H_p(K') \simeq H_p(K)$, *with any* g *giving the isomorphism via* g_*. *The analogous result holds for augmented chain complexes and reduced homology.*

Proof. The proof reduces to the case

$$K' \;=\; \mathrm{sd}(\mathrm{sd}(\ldots(\mathrm{sd}\,K)) \;=\; \mathrm{sd}^N K$$

We will explain the case $K' = \mathrm{sd}\,K$. Define λ as in the picture below
[Triangle with barycentric subdivision, τ's are the small triangles oriented anticlockwise]
$\lambda(\sigma) := \sum \tau_i$. Alternatively, we can proceed by induction on $\dim(\sigma)$

$$\lambda(\sigma) \quad `=' \quad [\hat{\sigma}\lambda\partial\sigma]$$

Why is it a chain map?

$$\partial\lambda(\sigma) \;=\; \lambda\partial\sigma - \partial\lambda\partial\sigma \;=\; \lambda\partial(\sigma)$$

so λ commutes with ∂. Need to show $\lambda \circ g_\#$ and $g_\# \circ \lambda$ are chain homotopic to respective identities. It suffices to use the acyclic carrier theorem on the two acyclic carriers, Φ from K to K

$$\Phi(\sigma) \;=\; \text{simplicial complex of } \sigma$$

and Ψ from K' to K',

$$\Psi(\tau) \;=\; K'(\sigma)$$

where σ is the smallest dimensional simplex of K' containing τ. Note that $K'(\sigma)$ is acyclic. □

11. Simplicial approximations

Definition 11.1. *Let v be a vertex of a simplicial complex K. The **star** of v, denoted $\mathrm{st}\,v$, s the union of $vv_1\ldots v_p - v_1\ldots v_p$ where $vv_1\ldots v_p$ are simplexes of K.*

Proposition 11.2. *We have $(\mathrm{st}\,v)^c = \bigcup_{\sigma\in K, v\notin\sigma}\sigma$. In particular, $\mathrm{st}\,v$ is open.*

Proof. Let $\sigma = a_0\ldots a_p$ be a simplex of K with where $a_i \neq v$ for all i. Let $vv_1\ldots v_q$ be a simiplex of K. So $vv_1\ldots v_q \cap a_0\ldots a_p$ is a face of $vv_1\ldots v_q$ which does not contain v. Hence

$$(vv_1\ldots v_q - v_1\ldots v_q)\cap a_0\ldots a_0 \;=\; \emptyset$$

so $\sigma \subseteq (\mathrm{st}\,v)^c$. But $\bigcup_{\sigma\in K, v\notin\sigma}\sigma \cup \mathrm{st}\,v = |K|$, hence $(\mathrm{st}\,v)^c = \bigcup_{\sigma\in K,v\notin\sigma}\sigma$ which is closed. □

Proposition 11.3. *Let K, L be simplicial complexes and $h : |K| \longrightarrow |L|$ be a continuous map. Suppose $f : K^{(0)} \longrightarrow L^{(0)}$ is a function satisfying*

($*$)**:** *for any $v \in K^{(0)}$, $h(\mathrm{st}\,v) \subseteq \mathrm{st}\,f(v)$ ($\subseteq |L|$).*

*Let $\sigma = a_0\ldots a_p$ be a simplex of K and $x \in \mathrm{Int}\,\sigma$. Pick a simplex τ of L such that $h(x) \in \mathrm{Int}\,\tau$. Then $f(a_i)$ are vertices of τ. In particular, f is a simplicial map $K \longrightarrow L$ and we call it a **simplicial approximation** to h.*

Proof. Since $x \in \bigcap \mathrm{st}\,a_i$, by the condition ($*$), $h(x) \in \bigcap \mathrm{st}\,f(a_i)$. If $f(a_i)$ is not a vertex of τ, then $\mathrm{st}\,f(a_i)\cap\tau = \emptyset$, but this contradicts $h(x) \in \mathrm{st}\,f(a_i)\cap\tau$. Hence $f(a_i)$ is a vertex of τ. □

Note 11.4. Suppose K is a subdivision of L and $f : K^{(0)} \longrightarrow L^{(0)}$ is a function satisfying the condition ($*$) above. Then f is a simplicial approximation to $\mathrm{id} : |K| \longrightarrow |L|$.

Lemma 11.5. *Let $f, g : K \longrightarrow L$ be a simplicial approximation to a continuous map $h : |K| \longrightarrow |L|$, then f, g are contiguous, so $f_* = g_* : H_p(K) \longrightarrow H_p(L)$.*

Proof. Let $\sigma = a_0\ldots a_p$, x and τ as in proposition 11.3. Then $f(a_0),\ldots,f(a_p),g(a_0),\ldots,g(a_p)$ span a face of τ, so f and g are contiguous. □

Lemma 11.6. *Let K, L, M be simplicial complexes and $h : |K| \longrightarrow |L|$, $h' : |L| \longrightarrow |M|$ be continuous maps. Let $f : K \longrightarrow L$ and $f' : L \longrightarrow M$ be simplicial approxmations of h and h'. Then $f'\circ f$ is a simplicial approximation to $h'\circ h$.*

Proof. Let $v \in K^{(0)}$, then $h(\mathrm{st}\,v) \subseteq \mathrm{st}\,f(v)$. Therefore $h'h(\mathrm{st}\,v) \subseteq h'(\mathrm{st}\,f(v)) \subseteq \mathrm{st}\,f'f(v)$. So $f'\circ f$ is a simplicial approximation to $h'\circ h$. □

11.1. **Simplicial approximation theorem.**

Theorem 11.7. *Let K, L be simplicial complexes and $h : |K| \longrightarrow |L|$ be a continuous function. For sufficiently large N, there exists a simplicial approximation $f : \mathrm{sd}^N K \longrightarrow L$ to $h : |\,\mathrm{sd}^N K| = |K| \longrightarrow |L|$.*

Proof. (Sketch) Note that $\{\mathrm{st}\, w\}_{w \in L^{(0)}}$ is an open cover of $|L|$, and we can pull this cover bac to $|K|$ by h. Claim: for sufficiently large N, given any vertex in $\mathrm{sd}^{\mathrm{NK}}$, $\mathrm{st}\, v$ is contained in some $h^{-1}(\mathrm{st}\, w)$. Then define $f : (\mathrm{sd}^N K)^{(0)} \longrightarrow L^{(0)}$ by $f(v) = w$ (any w as above), then f is a simplicial approximation of h. As $N \to \infty$, the diameter of the stars tend to zero. Now use Lebesgue number for covers. □

Lemma 11.8. *Given a compact metric space X and any cover $\{U_\alpha\}$, there is some $\varepsilon > 0$ such that for any set of diameter $< \varepsilon$ it is contained in one of the U_α's.*

Proof. Suppose $X_n \subseteq X$ with $\mathrm{diam}(X_n) < 1/n$, such that X_n is not contained in any U_α. Picking $x_n \in X_n$ we can find a limit point $x \in X$ of $\{x_n\}$ since X is compact. □

12. Topological invariance

Recall that H_p is a functor $\mathfrak{SimpS} \longrightarrow \mathfrak{Grp}$. Define a new category $\mathfrak{SimpC}$ with the same objects as $\mathfrak{SimpS}$ but with morphisms as continuous maps of polytopes, that is, $\mathrm{Hom}_{\mathfrak{SimpC}}(K, L) = \mathrm{Hom}_{\mathfrak{Top}}(|K|, |L|)$.

Theorem 12.1. *For each $p \in \mathbb{Z}$ we have a functor $H_p : \mathfrak{SimpC} \longrightarrow \mathfrak{Grp}$.*

Proof. Firstly, H_p is defined on objects since $\mathrm{Ob}(\mathfrak{SimpC}) = \mathrm{Ob}(\mathfrak{SimpS})$. Next we need to define H_p on morphisms. Let K and L be simplicial complexes and $h : |K| \longrightarrow |L|$ be continuous. We need a group morphism $H_p(h) : H_p(K) \longrightarrow H_p(L)$. Pick a subdivision K_1 of K so there is a simplicial approximation $f : K_1 \longrightarrow L$ for $h : |K_1| \longrightarrow |K|$; and $g : K_1 \longrightarrow K$ for $\mathrm{id} : |K_1| \longrightarrow |K|$. Recall that $g_* = H_p(g)$ is an isomorphism. Define $H_p(h)$ to be the composition

$$H_p(K) \xrightarrow{g_*^{-1}} H_p(K_1) \xrightarrow{f_*} H_p(L).$$

For fixed K_1, $H_p(h)$ is independent of the choice of simplicial approximations f (and g), since any two such choices are contiguous by lemma 11.5. Next we need to show $H_p(h)$ is independent of the choice of subdivision K_1. Suppose K_1' is a subdivision of K and simplicial approximations $f' : K_1' \longrightarrow L$ for h; and $g' : K_1' \longrightarrow K$ for the identity. Let K_2 be a subdivision of K_1 such that we have simplicial approximations $g_2' : K_2 \longrightarrow K_1'$ for $\mathrm{id} : |K_2| \longrightarrow |K_1'|$ and $g_2 : K_2 \longrightarrow K_1$ for $\mathrm{id} : |K_2| \longrightarrow |K_1|$,

$$\begin{array}{ccccc} & & K_1 & & \\ & {\scriptstyle g_2} \nearrow & {\scriptstyle g} \downarrow & \searrow {\scriptstyle f} & \\ K_2 & & K & & L \\ & {\scriptstyle g_2'} \searrow & {\scriptstyle g'} \uparrow & \nearrow {\scriptstyle f'} & \\ & & K_1' & & \end{array}$$

Note that fg_2 and $f'g_2'$ are simplicial approximations to $h \circ \mathrm{id} = h$, by lemma 11.5 they are contiguous. Similarly gg_2 and $g'g_2'$ are also contiguous. Using these contiguities, we have

$$\begin{aligned} f'_* g'^{\,-1}_* &= (f'_* g'_{2,*})(g'^{\,-1}_{2,*} g'^{\,-1}_*) \\ &= f_* g_{2,*} g_{2,*}^{-1} g_*^{-1} \\ &= f_* g_*^{-1} \end{aligned}$$

so the map is independent on choice of K_1.
Now since $\mathrm{id} : K \longrightarrow K$ is already a simplicial map, we have $H_p(\mathrm{id}) = \mathrm{id}$. Finally we have to check that $H_p(h \circ h') = H_p(h) \circ H_p(h')$. Let $|K| \xrightarrow{h'} |L| \xrightarrow{h} |M|$ and consider the simplicial approximations

$$\begin{array}{ccccc} K_1 & \xrightarrow{f'} & L_1 & \xrightarrow{f} & M \\ {\scriptstyle g'} \downarrow & & {\scriptstyle g} \downarrow & & \\ K & & L & & \end{array}$$

where f approximates h; f' approximates h'; g and g' approximate the identity. So ff' approximates hh'; gf' approximates h'.

$$H_p(h) \circ H_p(h') = f_* g_*^{-1} g_* f'_* g'^{\,-1}_* = f_* f'_* g'^{\,-1}_* = H_p(h \circ h')$$

This completes the proof of the theorem. □

Corollary 12.2. *Let $K \longrightarrow L$ be simplicial complexes and $h : |K| \longrightarrow |L|$ is a homeomorphism. Then $H_p(h)$ is an isomorphism.*

Proof. Since h is an isomorphism in $\mathfrak{SimpC}$ and functors take isomorphisms to isomorphisms so by theorem 12.1 $H_p(h)$ is an isomorphism. □

12.1. **Functoriality.** Let $\mathfrak{TrTop}$ be the category of topological spaces which have a triangulation, with morphisms as continuous maps.

Theorem 12.3. *For each $p \in \mathbb{Z}$ there is a functor $H_p : \mathfrak{TrTop} \longrightarrow \mathfrak{Grp}$.*

Proof. (Sketch.) Let $X \in \mathrm{Ob}(\mathfrak{TrTop})$. Consider all triangulations $h_\alpha : |K_\alpha| \longrightarrow X$. By corollary 12.2, all $H_p(K_\alpha)$ are isomorphic, essentially, $H_p(X)$"="$H_p(K_\alpha)$ for any α. Technically

$$H_p(X) \quad := \quad \frac{\dot{\bigcup} H_p(K_\alpha)}{\sim}$$

where for β, β', $H_p(K_\beta) \overset{h_{\beta',*} \circ h_\beta^{-1}}{\longrightarrow} H_p(K_{\beta'})$. Each $H_p(K_\alpha)$ induces the same group structure on $H_p(X)$, similarly define H_p on morphisms. □

13. Homotopy invariance

Endow $I = [0, 1]$ with the Euclidean topology.

Definition 13.1. *Let X, Y be topological spaces. Two continuous maps $f, g : X \longrightarrow Y$ are **homotopic**, denoted $f \simeq g$, if there is a continuous map, called a **homotopy**, $h : X \times I \longrightarrow Y$ such that $h(x, 0) = f(x)$, $h(x, 1) = g(x)$ for all $x \in X$.*

In this case define $h_t := h(-, t) : X \longrightarrow Y$, giving a continuous family of continuous maps $\{h_t\}_{t \in I}$ which exhibits a continuous deformation of $h_0 = f$ to $h_1 = g$.

Example 13.2. Let $X = \mathbb{R}^n - \{0\}$, $f = \mathrm{id}_X$, $g : X \longrightarrow X$ given by $x \longmapsto x/\|x\|$. These maps are homotopic, since we have a homotopy

$$\begin{aligned} h : X \times I &\longrightarrow X \\ (x, t) &\longmapsto \frac{x}{\|x\|}(\|x\|(1-t) + t)\,. \end{aligned}$$

Lemma 13.3.

(1) The relation $\simeq$ is an equivalence relation.
(2) If $f \simeq g$ then $pf \simeq pg$, $fq \simeq gq$ whenever p and q are continuous maps such that both sides are defined.

Proof. Only do some checks. Let h be a homotopy between f and g such that $h(-, 0) = f$ and $h(-, 1) = g$. Then $h'(-, t) := h(-, 1-t)$ is a homotopy from g to f, so $\simeq$ is symmetric. Let $p : Y \longrightarrow Z$ be a continuous map, then

$$\begin{aligned} p \circ h : X \times I &\longrightarrow Z \\ (x, t) &\longmapsto ph(x, t) \end{aligned}$$

is a homotopy between pf and pg. □

Definition 13.4. *There is a well defined **homotopy category** (of topological spaces) H, with $\mathrm{Ob}(H) = \mathrm{Ob}(\mathfrak{Top})$ and $\mathrm{Hom}_H(X, Y) = \mathrm{Hom}_{\mathfrak{Top}}(X, Y)/\simeq$. If $f : X \longrightarrow Y$, $g : Y \longrightarrow Z$ are continuous and $[f]$, $[g]$ denote their homotopy equivalence classes, then the above lemma allows us to define $[g] \circ [f] := [g \circ f]$.*

Definition 13.5. *Let $f : X \longrightarrow Y$ be a continuous map. It is a **homotopy equivalence** if $[f]$ is an isomorphism in H. A continuous map $g : Y \longrightarrow X$ is a **homotopy inverse** if $[g] = [f]^{-1}$.*

Example 13.6. The maps $f : \mathbb{R}^n - \{0\} \longrightarrow S^{n-1}$ given by $x \longmapsto x/\|x\|$ and $\iota : S^{n-1} \longrightarrow \mathbb{R}^n - \{0\}$ given by inclusion are homotopy inverses. $f \circ \iota = \mathrm{id}_{S^{n-1}}$, $\iota \circ f \simeq \mathrm{id}_{\mathbb{R}^n - \{0\}}$ by example 13.2.

Theorem 13.7.

(1) Let X, Y be topological spaces with triangulations and $f, g : X \longrightarrow Y$ be homotopic continuous maps. Then they induce the same maps in homology and reduced homology.
(2) Let $\mathrm{Tr}\, H$ be the homotopy category of triangulable topological spaces with $\mathrm{ob}(\mathrm{Tr}\, H) = \mathrm{Ob}(\mathrm{Tr}\, \mathfrak{Top})$, $\mathrm{Hom}(\mathrm{Tr}\, H) =$homotopy equivalence classes of continuous maps. Then the functor $H_p : \mathrm{Tr}\, \mathfrak{Top} \longrightarrow \mathfrak{Grp}$ induces a functor $H_p : \mathrm{Tr}\, H \longrightarrow \mathfrak{Grp}$ and similarly for $\tilde{H}_p$.

Proof. (Sketch.) It is clear that $1 \Longrightarrow 2$. Suppose $X = |K|$ and $Y = |L|$ and $K \times I \longrightarrow L$, $|K| \times I \longrightarrow |L|$, i_0 and i_1 are simplicial approximations to h. It suffices to show $i_{0,\#}$ and $i_{1,\#}$ are chain homotopic using acyclic carrier theorem. □

Theorem 13.8. *The topological spaces $\mathbb{R}^n$ and $\mathbb{R}^m$ are not homeomorphic unless $n = m$.*

Proof. Suppose $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is a homeomorphism, which restricts to a homeomorphism $\mathbb{R}^n - \{0\} \longrightarrow \mathbb{R}^m - \{0\}$, so these are homotopy equivalent. But $\mathbb{R}^n - \{0\}$ is homotopy equivalent to S^{n-1}. Being isomorphic in a category is an equivalence relation so S^{n-1} and S^{m-1} are homotopy equivalent, hence has isomorphic homology. However, the homology of S^{n-1} is concentrated in degrees $n-1$ and 0, so $\mathbb{R}^n$ and $\mathbb{R}^m$ cannot be homeomorphic unless $n = m$. □

14. Exactness and fractions

Let $\mathcal{C}$ be a module category, for example $\mathcal{C} = \mathfrak{Ab}$, k **-Mod**, R **-Mod**.

Definition 14.1. *A sequence* $\dots \longrightarrow C_{i+1} \xrightarrow{\partial_{i+1}} C_i \longrightarrow C_{i-1} \longrightarrow \dots$ *in* $\mathcal{C}$ *is **exact at*** C_i *if* $\ker(\partial_i) = \text{im}(\partial_{i+1})$. *It is **exact** if it is exact at every* C_i.

Since exact sequences are essentially acyclic complex, we can talk about morphisms of exact sequences. Special cases: $0 \longrightarrow A' \xrightarrow{i} A$ is exact means i is injective, $A \xrightarrow{\pi} A'' \longrightarrow 0$ means π is surjective, $0 \longrightarrow A \xrightarrow{f} B \longrightarrow 0$ means f is an isomorphism, and a **short exact sequence** is an exact sequence of the form $0 \longrightarrow A' \xrightarrow{f} A \xrightarrow{g} A'' \longrightarrow 0$. By the first isomorphism this means f is injective and g induces an isomorphism $\simeq B/f(A)$.

Definition 14.2. *Let* $\mathcal{C}$ *and* $\mathcal{D}$ *be module categories. A functor* $F : \mathcal{C} \longrightarrow \mathcal{D}$ *is exact for every exact sequence* $A \longrightarrow B \longrightarrow C$ *in* $\mathcal{C}$*, the sequence* $F(A) \longrightarrow F(B) \longrightarrow F(C)$ *is exact in* $\mathcal{D}$.

14.1. **Fractions.** Let A be an abelian group. Let $A_{\mathbb{Q}}$ =set of equivalence classes of pairs a/n where $a \in A$ and $n \in \mathbb{Z} - \{0\}$ and $a/n \sim a'/n'$ iff $mn'a = mna'$ for some $m \in \mathbb{Z} - \{0\}$. This is an equivalence relation. Note that $a/n \sim pa/pn$ for $p \in \mathbb{Z} - \{0\}$ so we can find a common denominator for any two fractions.

Proposition 14.3. $A_{\mathbb{Q}}$ *is a* $\mathbb{Q}$*-vector space with addition defined by* $a/n + a'/n' = \frac{n'a + na'}{nn'}$*, and scalar multiplication defined* $(p/q)(a/n) = pa/qn$ *for* $p/q \in \mathbb{Q}$.

Proof. Omitted. Note that $a/n + a'/n = n(a + a')/n^2 \sim (a + a')/n$. □

Example 14.4. The map $(\mathbb{Z}^r)_{\mathbb{Q}} \longrightarrow \mathbb{Q}^r$ given by $(a_1, \dots, a_r)/n \longmapsto (a_1/n, \dots, a_r/n)$ is an isomorphism. Exercise: show that φ is linear.

Example 14.5. Let A =finite abelian group, then $A_{\mathbb{Q}} = 0$. Let $a/n \in A_{\mathbb{Q}}$, and m be the order of A, then $ma = 0$, hence $a/n \sim 0/1$.

Example 14.6. Show that $(\mathbb{Z}^r \times A)_{\mathbb{Q}} = \mathbb{Q}^r$ if A is finite abelian.

Proposition 14.7. *Let* A *and* B *be a group morphism.*

1. *Then there is a well defined* $\mathbb{Q}$*-linear map* $f_{\mathbb{Q}} : A_{\mathbb{Q}} \longrightarrow B_{\mathbb{Q}}$ *given by* $a/n \longmapsto f(a)/n$.
2. $(-)_{\mathbb{Q}} : \mathfrak{Ab} \longrightarrow \mathbb{Q}$ ***-Mod*** *is a functor.*

15. Exact functors and trace

Lemma 15.1. *Let* $F : \mathcal{C} \longrightarrow \mathcal{D}$ *be a functor between module categories such that* $F(0) = 0$. *Then* F *induces a functor* $C(\mathcal{C}) \longrightarrow C(\mathcal{D})$.

Proof. Let $(C_\bullet, \partial)$ be a complex in $\mathcal{C}$, then $F(\partial)^2 = F(d^2) = F(0) = 0$ since the zero morphism factors through the 0 object, $A \longrightarrow 0 \longrightarrow B$. □

Theorem 15.2. *The functor* $- \otimes_{\mathbb{Z}} \mathbb{Q} = (-)_{\mathbb{Q}} : \mathfrak{Ab} \longrightarrow \mathbb{Q}$ ***-Mod*** *is exact.*

Proof. Let $A' \xrightarrow{f} A \xrightarrow{g} A''$ be exact in $\mathfrak{Ab}$. Since $(0)_{\mathbb{Q}} = 0$ $(A')_{\mathbb{Q}} \longrightarrow (A)_{\mathbb{Q}} \longrightarrow (A'')_{\mathbb{Q}}$ is a complex. Suppose $b/n \in \ker(g_{\mathbb{Q}})$, that is, $mg(b) = g(mb) = 0$ for some $m \in \mathbb{Z} - \{0\}$. So $mb \in \ker(g)$, and by exactness $x = f^{-1}(mb)$ exists. Now $f_{\mathbb{Q}}(x/m) = f(x)/nm = b/n$ so $(-)_{\mathbb{Q}}$ is exact. □

15.1. **Exactness preserves homology.**

Proposition 15.3. *Let* $F : \mathcal{C} \longrightarrow \mathcal{D}$ *be an exact functor.*

1. $F(0) = 0$, *so* F *takes complexes to complexes.*
2. *Let* $C_\bullet$ *be a complex in* $\mathcal{C}$*, then* $F(H_k(C_\bullet)) \simeq H_k(F(C_\bullet))$ *for all* k.

Proof. Apply F to the zero sequence $0 \longrightarrow 0 \longrightarrow 0$ and use exactness.
Let $C_0 \xrightarrow{f} C_1 \xrightarrow{g} C_2$ exact in $\mathcal{C}$, then by exactness of F, $\ker(F(g)) \simeq F(\ker(g))$ and $\text{im}(F(f)) \simeq F(\text{im}(f))$. Applying F to $0 \longrightarrow \text{im}(f) \longrightarrow \ker(g) \longrightarrow H_1(C_\bullet) \longrightarrow 0$ so by exactness

$$F(H_1(C_\bullet)) \simeq \frac{F(\ker(g))}{F(\text{im}(f))} \simeq \frac{\ker(F(g))}{\text{im}(F(f))} \simeq H_1(F(C_\bullet)).$$

□

15.2. **Trace.** Let k be a field and $V = k^n$, let $\varphi : V \longrightarrow V$ be a linear map. Pick an ordered basis B for V and suppose A is an $n \times n$-matrix representing φ w.r.t. B.

Proposition 15.4. *The* ***trace*** *of φ is $\mathrm{tr}(\varphi; V) := \mathrm{tr}(A)$. This is independent of the choice of B.*

Proof. Exercise. □

15.3. **Additivity of trace.**

Proposition 15.5. *Let $0 \longrightarrow V' \xrightarrow{i} V \xrightarrow{\pi} V'' \longrightarrow 0$ be an exact sequence of k-vector spaces, and let φ be an endomorphism of the above sequence. Then $\mathrm{tr}(\varphi) = \mathrm{tr}(\varphi') + \mathrm{tr}(\varphi'')$.*

Proof. The sequence is split, that is $V \simeq V' \oplus V''$, so pick bases B' and B'' for V' and V''. This gives a basis for V under the splitting isomorphism, and respect to this basis,

$$\varphi = \begin{pmatrix} \varphi' & * \\ 0 & \varphi'' \end{pmatrix}$$

hence the result. □

16. Hopf trace formula and Lefschetz fixed point theorem

Let $X \in \mathfrak{TrTop}$ and $\varphi : X \longrightarrow X$ be a continuous map. The **Lefschetz number** of φ is defined as

$$\Lambda(\varphi) = \sum_{p \in \mathbb{Z}} (-1)^p \ \mathrm{tr}\left((\varphi_*)_{\mathbb{Q}}, H_p(X)_{\mathbb{Q}}\right)$$

Setting $\varphi = \mathrm{id}_X$ gives the Euler characteristic of X

$$\chi(X) = \Lambda(\mathrm{id}_X) = \sum_{p \in \mathbb{Z}} (-1)^p h_p(X)$$

where $h_p(X) = \dim_{\mathbb{Q}}(H_p(X) \otimes_{\mathbb{Z}} \mathbb{Q}) = p$-th Betti number. Note that $\Lambda(\varphi)$, hence $\chi(X)$, depends only on the homotopy class of φ.

Theorem 16.1. *Let $C_\bullet$ be a complex of vector spaces over a field k and $\varphi_\bullet : C_\bullet \longrightarrow C_\bullet$ be a morphism of complexes. Then*

$$\sum_{p \in \mathbb{Z}} (-1)^p \ \mathrm{tr}(H_p(\varphi), H_p(C_\bullet)) = \sum_{p \in \mathbb{Z}} (-1)^p \ \mathrm{tr}(\varphi_p, C_p)$$

Proof. Consider exact sequences

$$0 \longrightarrow B_p(C_\bullet) \longrightarrow Z_p(C_\bullet) \longrightarrow H_p(C_\bullet) \longrightarrow 0$$

and

$$0 \longrightarrow Z_p(C_\bullet) \longrightarrow C_p(C_\bullet) \xrightarrow{\partial} B_{p-1}(C_\bullet) \longrightarrow 0.$$

Dropping the $C_\bullet$, we have

$$\begin{aligned} \sum_{p \in \mathbb{Z}} (-1)^p \ \mathrm{tr}(\varphi, Z_p) &= \sum_{p \in \mathbb{Z}} (-1)^p \ \mathrm{tr}(\varphi, B_p) + \sum_{p \in \mathbb{Z}} (-1)^p \ \mathrm{tr}(H_p(\varphi), H_p) \\ \sum_{p \in \mathbb{Z}} (-1)^p \ \mathrm{tr}(\varphi, C_p) &= \sum_{p \in \mathbb{Z}} (-1)^p \ \mathrm{tr}(\varphi, Z_p) + \sum_{p \in \mathbb{Z}} (-1)^p \ \mathrm{tr}(\varphi, B_{p-1}) \end{aligned}$$

This gives the formula. □

Corollary 16.2. *Let K be a simplicial complex, then*

$$\chi(|K|) = \sum_{k \in \mathbb{Z}} (-1)^k \textit{number of } k\textit{-simplexes}$$

depends only on the homotopy class of K.

Proof. Apply the Hopf trace formula to $C_\bullet(K)_{\mathbb{Q}}$ and note that $H_p(C_\bullet(K)_{\mathbb{Q}} = H_p(C_\bullet)_{\mathbb{Q}}$ since $(-)_{\mathbb{Q}}$ is exact so preserves homology. □

16.1. **Lefschetz fixed point theroem.**

Theorem 16.3. *Let $X \in \mathfrak{TrTop}$ and $\varphi : X \longrightarrow X$ be continuous. If $\Lambda(\varphi) \neq 0$, then φ has a fixed point.*

Proof. (Sketch.) Consider a triangulation $|K| \longrightarrow X$ of X. Let $f : \operatorname{sd}^N K \longrightarrow K$ be a simplicial approximation for φ and a simplicial approximation to id_X, $g : \operatorname{sd}^N \longrightarrow K$. Let $\lambda_\bullet : C_\bullet(K) \longrightarrow C_\bullet(\operatorname{sd}^N K)$ be the chain homotopy inverse to $g_\#$ constructed in theorem 10.4. Assume there is no fixed point, then by the Hopf trace formula it suffices to show that the alternating sum of $\operatorname{tr}(\varphi_p, C_p(K)_\mathbb{Q})$ is zero, where

$$\begin{array}{ccc} \varphi_p : C_p(K)_\mathbb{Q} & \longrightarrow & C_p(K) \\ & \lambda_p \searrow \nearrow f_\# & \\ & C_p(\operatorname{sd}^N K) & \end{array}$$

Unproved fact: For K sufficiently subdivided, we have: for any oriented p-simplex σ in K, the coefficient of σ in $f_\# \circ \lambda_p(\sigma)$ is zero.
We will compute $\operatorname{tr}(\varphi_p, C_p(K)_\mathbb{Q}$. Pick the usual basis for $C_p(K)_\mathbb{Q}$ consisting of appropriately oriented p-simplexes $\{[\sigma]\}$. Let A_p represent φ_p with respect to this bases. The unproved fact tells us that the diagonal entries of A_p is zero, hence $\operatorname{tr}(A_p) = 0$. □

Corollary 16.4. *Let X be a triangulable space such that X is connected and $H_p(X) = 0$ for $p > 0$, e.g. X is homotopy equivalent to a ball or cone. Let $\varphi : X \longrightarrow X$ be a continuous map, then φ has a fixed point.*

Proof. Simply note that $\Lambda(\varphi) = \operatorname{tr}(\varphi_0, H_0(X)) = 1$ since X is connected, hence the result follows by the Lefschetz fixed point theorem. □

17. Long exact sequence in homology

Let $\mathcal{C}$ be a module category.

Definition 17.1. *A **short exact sequence of (chain) complexes** in $\mathcal{C}$ is a sequence of chain maps*

$$0 \longrightarrow C'_\bullet \xrightarrow{f_\bullet} C_\bullet \xrightarrow{g_\bullet} C''_\bullet \longrightarrow 0$$

such that for each p, $0 \longrightarrow C'_p \xrightarrow{f_p} C_p \xrightarrow{g_p} C''_p \longrightarrow 0$ is a short exact sequence in $\mathcal{C}$.

Example 17.2. Let K be a simplicial complex and L a subcomplex, we get a short exact sequene of complexes,

$$0 \longrightarrow C_\bullet(L) \longrightarrow C_\bullet(K) \longrightarrow C_\bullet(K, L) = C_\bullet(K)/C_\bullet(L).$$

Theorem 17.3. *Consider a short exact sequence of chain complexes $0 \longrightarrow C'_\bullet \xrightarrow{f_\bullet} C_\bullet \xrightarrow{g_\bullet} C''_\bullet \longrightarrow 0$ in $\mathcal{C}$, then there exists a long exact sequence*

$$\ldots \longrightarrow H^k(C_\bullet) \longrightarrow H^k(C''_\bullet) \xrightarrow{\Delta_p} H^{k-1}(C'_\bullet) \longrightarrow \ldots$$

in homology, where Δ_p is a morphism in $\mathcal{C}$ called the connecting homomorphism.

Proof. Denote H'_p for $H_p(C'_\bullet)$ and $Z''_{p-1} = Z_{p-1}(C''_\bullet)$ etc. First we define the connecting homomorphism Δ_p. Let $\sigma \in Z''_p$, then choose $\tau \in g_p^{-1}(\sigma)$, and consider $\tau' := f_{p-1}^{-1}(\partial\tau) \in C'_{p-1}$. In fact, $\tau' \in Z'_{p-1}$ since $\partial\tau' = 0$ iff $f_{p-2}\partial\tau' = 0$ and $f_{p-2}\partial\tau' = \partial f_{p-1}\tau' = \partial^2\tau = 0$, so we can define $\Delta_p([\sigma]) = [\partial\tau'] \in H'_{p-1}$. We check this definition does not depend on the choice of σ and τ.
Suppose $\sigma' \in Z''_p$ such that $\sigma - \sigma' \in B''_p$, so let $\rho'' \in C''_{p+1}$ such that $\partial\rho'' = \sigma - \sigma'$. Let $\rho \in g_{p+1}^{-1}(\rho'')$, then τ changes by $\partial\rho''$, so $[\partial\tau']$ does not change. Let τ' be another element in $g_p^{-1}(\sigma)$, then $\tau - \tau' \in \ker(g_p) = \operatorname{im}(f_p)$, so let $\omega \in f_p^{-1}(\tau - \tau')$ and $[\partial\tau']$ changes by a boundary, namely $\partial\omega$. Hence Δ_p is independent of choices made.
Exercise: check that Δ_p is a morphism in $\mathcal{C}$.
Note that H_p is functorial, so $\ker(H_p(g_\bullet)) \supseteq \operatorname{im}(H_p(f_\bullet))$.
Exercise: check exactness. □

18. The Mayer-Vietoris sequence

Definition 18.1. *Let $\mathcal{C}$ be a module category and $C_\bullet, C'_\bullet \in C(\mathcal{C})$. The category $C(\mathcal{C})$ admits **direct sums**, given by*

$$(C_\bullet \oplus C'_\bullet)_p \quad = \quad C_p \oplus C'_p$$

with boundary maps given by $\partial_p = (\partial, \partial')$. Moreover $H_p(C_\bullet \oplus C'_\bullet) = H_p(C_\bullet) \oplus H_p(C'_\bullet)$.

Theorem 18.2. *Let K be a simplicial complex and K_1, K_2 be subcomplexes. Denote $K_0 = K_1 \cap K_2$; this is a subcomplex. If $K = K_1 \cup K_2$, then there is a long exact sequence in homology,*

$$\dots \longrightarrow H_p(K_0) \longrightarrow H_p(K_1) \oplus H_p(K_2) \longrightarrow H_p(K) \longrightarrow H_{p-1}(K_0) \longrightarrow \dots$$

This is known as the ***Mayer-Vietoris sequence****. There exists a similar sequence in reduced homology if K_0 is nonempty.*

Proof. It suffices to construct a short exact sequence

$$0 \longrightarrow C_\bullet(K_0) \xrightarrow{f_\bullet} C_\bullet(K_1) \oplus C_\bullet(K_2) \xrightarrow{g_\bullet} C_\bullet(K) \longrightarrow 0$$

and we obtain the Mayer-Vietoris sequence by the long exact sequence in homology. We have inclusion maps

$$\begin{array}{ccc} & C_\bullet(K_1) & \\ & \nearrow\searrow & \\ C_\bullet(K_0) & \longrightarrow & C_\bullet(K) \\ & \searrow\nearrow & \\ & C_\bullet(K_2) & \end{array}$$

Define maps

$$\begin{array}{rcl} f_\bullet : C_\bullet(K_0) & \longrightarrow & C_\bullet(K_1) \oplus C_\bullet(K_2) \\ c & \longmapsto & (c, -c) \\ C_\bullet(K_1) \oplus C_\bullet(K_2) & \longrightarrow & C_\bullet(K) \\ g_\bullet : (c_1, c_2) & \longmapsto & (c_1, c_2) \end{array}$$

Firstly f_p is a homomorphism, since $f_p(c + c') = (c + c', -c - c') = f_p(c) + f_p(c')$; secondly f and ∂ commute $\partial f(c) = \partial(c, -c) = (\partial c, -\partial c) = f\partial(c)$, so $f_\bullet$ is a chain map. Similarly, $g_\bullet$ is also a chain map. Clearly $g_p \circ f_p = 0$ for all p, f_p is injective for all p, and g_p is surjective since $K = K_1 \cup K_2$. Finally, let $(c, c') \in \ker(g_p)$, then $c + c' = 0$ so $f(c) = (c, c')$. □

Example 18.3. The Mayer-Vietoris sequence can often be used to compute $H_p(S^1 \times X)_{\mathbb{Q}}$ from $H_p(X)_{\mathbb{Q}}$. To illustrate this, we will compute $H_\bullet(\mathbb{T}^2)$.

We can consider $\mathbb{T}^2$ as a union of two cylinders, K_1 and K_2 whose intersection K_0 consists of two disjoint circles, K_0' and K_0'. Let γ', γ'' be a basis for $C_1(K_0)$, and v', v'' be a basis for $C_0(K_0)$. In homology

$$\begin{array}{rcl} H_0(K_0) & = & \mathbb{Z}v' \oplus \mathbb{Z}v'' \\ H_0(K_0) & = & \mathbb{Z}\gamma' \oplus \mathbb{Z}\gamma''. \end{array}$$

Now the cylinder is homotopic to the circle, hence $\mathbb{Z}\gamma'' \simeq H_1(K_1) \overset{\text{homotopy}}{\longrightarrow} H_1(K_{0,1}) \simeq \mathbb{Z}\gamma'$. In fact, γ' and γ'' correspond naturally, so $H_1(K_1) \simeq \mathbb{Z}\gamma_1$ where γ_1 corresponds to γ' or γ''. Similarly $H_1(K_2) = \mathbb{Z}\gamma_2$ where γ_2 corresponds to γ' or γ''. Now $(-)_{\mathbb{Q}}$ is exact, and the cohomology K_1, K_2 vanishes in degree 2, so we have

$$0 \longrightarrow H_2(\mathbb{T}^2)_{\mathbb{Q}} \longrightarrow H_1(K_0)_{\mathbb{Q}} \xrightarrow{f_{1,*}} (H_1(K_1) \oplus H_1(K_2))_{\mathbb{Q}} \xrightarrow{g_{1,*}} H_1(\mathbb{T}^2)_{\mathbb{Q}} \xrightarrow{\Delta_1} H_0(K_0)_{\mathbb{Q}} \xrightarrow{f_{0,*}} H_0(K_1) \oplus H_0(K_2)$$

where the maps are as follows

$$\begin{array}{rcl} f_{1,*}(m\gamma' + n\gamma'') & \longmapsto & ((m+n)\gamma_1, -(m+n)\gamma_2) \\ f_{0,*}(mv' + nv'') & \longmapsto & (m+n, -(m+n)) \end{array}$$

Firstly $H_2(\mathbb{T}^2)_{\mathbb{Q}} = \ker(f_{1,*}) = \operatorname{span}(\gamma' - \gamma'') \simeq \mathbb{Q}$. To find $H_1(\mathbb{T}^2)_{\mathbb{Q}}$:

$$\begin{array}{rcl} \operatorname{rank}(\Delta_1) & = & \dim(\operatorname{im}(\Delta_1)) \\ & = & \dim(\ker(f_{0,*})) \\ \operatorname{nullity}(\Delta_1) & = & \operatorname{rank}(g_{1,*}) \\ & = & \dim(\mathbb{Q}\gamma_1 \oplus \mathbb{Q}\gamma_2) - \operatorname{nullity}(g_{1,*}) \\ & = & 2 - \operatorname{rank}(f_{1,*}) \; = \; 1 \end{array}$$

The rank-nullity theorem gives $H_1(\mathbb{T}^2)_{\mathbb{Q}} = 1 + 1 = 2$, which agrees with the previous calculuation of the first Betti number.

Lemma 18.4. *Let $I = [0, 1]$ and K be a simplicial complex. Then there is a simplicial complex $K \times I$ whose polytope $|K| \times I$, and such that $|K| \times 0$ and $|K| \times 1$ are triangulated as in K.*

19. Revision of Modules

Let R be a ring (in this course, R will be associative with identity).

Definition 19.1. *A (left)* ***R -module*** *M is an abelian group equipped with a multiplication map $\mu : R \times M \to M$ (given by $(r,m) \to \mu(r,m) =: rm$) such that for all $r, r' \in R$ and $m, m' \in M$:*

(1) $1 \cdot m = m$
(2) $(rr')m = r(r'm)$
(3) $(r + r')m = rm + r'm$
(4) $r(m + m') = rm + rm'$

Example 19.2. $R = \mathbb{Z}$. A $\mathbb{Z}$-module is essentially just an abelian group, because in abelian groups there is a unique way to take scalar multiples by integers.

Proposition 19.3. *Let M, N be R-modules. An* ***R -module homomorphism*** *$\phi : M \to N$ is a group homomorphism satisfying $\phi(rm) = r\phi(m)$ for all $r \in R$, $m \in M$. Let R-Mod denote the category of R-modules (with R-module homomorphisms).*

Example 19.4. $\mathbb{Z}$-Mod '=' $\mathfrak{Ab}$, the category of abelian groups.

Example 19.5. $M = R$ is a left R-module with μ being ring multiplication.

Proposition 19.6. *Let $M \in R$-Mod.*

(1) A ***submodule*** *N of M is a subgroup with $rn \in N$ for all $r \in R$. A submodule is an R-module and we write $N \leq M$.*
(2) Let N be a submodule of M, then M/N has a natural R-module structure $r(m + N) := rm + N$ with the quotient map $m \to m + N$ ($M \to M/N$) an R-module homomorphism.
(3) Let $\phi : M \to N$ be an R-module homomorphism. Then $\ker(\phi)$ is a submodule of M and $\mathrm{im}(\phi)$ is a submodule of N.

Complexes in R-Mod can be defined, since there's a 0 R-module, and for any two R-modules M, N there is a 0 R-module homomorphism $M \to N$ (taking $m \to 0$). By Proposition-Definition 2 above, exactness can be defined, and homology of a complex in R-Mod is an R-module.

Example 19.7. Let $x \in R$ and let

$$C_\bullet : 0 \longrightarrow R \xrightarrow{\cdot x} R \longrightarrow 0$$

be a complex of R-modules. The homology is concentrated in degrees 0 and 1

$$H_i(C_\bullet) = \begin{cases} R/Rx & \text{if } i = 0 \\ \mathrm{ann}_R(x) & \text{if } i = 1 \end{cases}$$

where $\mathrm{ann}_R(x) = \{r \mid rx = 0\}$.

19.1. Isomorphism Theorems.

Theorem 19.8. *Let $\phi : M \to N$ be an R-module homomorphism. If $M' \leq \ker(\phi)$ then $\psi : M/M' \to N$ (taking $m + M' \to \phi(m)$) is a well-defined R-module homomorphism. Setting $M' = \ker(\phi)$ gives $M/\ker(\phi) \simeq \mathrm{im}(\phi)$.*

Theorem 19.9. *Given R-modules $M'' \leq M' \leq M$ we have $M'/M'' \leq M/M''$ and $\frac{M/M''}{M'/M''} \simeq M/M'$.*

Theorem 19.10. *For R-modules $H, N \leq M$ we have $(H + N)/N \simeq H/(H \cap N)$.*

19.2. **Direct Sums.** Let $\{M_i\}_{i \in I}$ be a set of R-modules indexed by I. There is an R-module $\bigoplus_{i \in I} M_i$ whose elements are formal finite sums $\sum_{i \in I} m_i$ where $m_i \in M_i$ for all $i \in I$ and only finitely many of the m_i are non-zero. Define addition by $\sum m_i + \sum m_i' := \sum (m_i + m_i')$ and multiplication by $r \sum m_i := \sum (rm_i)$.

Note 19.11. For any $j \in I$, then M_j is a submodule of $\bigoplus_{i \in I} M_i$.

19.3. **Universal Property.** Let $M \in R$-Mod and $\{M_i\}$ be as above. Given an R-module homomorphism $\phi_i : M_i \to M$ for each $i \in I$ then $\phi : \bigoplus_{i \in I} M_i \to M$ defined by $\sum_{i \in I} m_i \to \sum \phi_i(m_i)$ is a well-defined R-module homomorphism.

19.4. **Free Modules.** A free R-module is an R-module which is isomorphic to a direct sum of copies of R. Suppose M is free and $M \simeq \bigoplus_{i \in I} R_i$ where each R_i is a copy of R.
Let $e_j \in \bigoplus_{i \in I} R_i$ where e_j is the image of 1 under the natural inclusion into the j-th component, $R \xrightarrow{\iota_j} \bigoplus_{i \in I} R_i$. The subset $\{e_j\}_{j \in I}$ of M is called a **basis** for M.

Remark 19.12. If R is a field, any R-module (vector space) is free (and has a basis).

20. Homology with coefficients

Let R be a commutative ring and K a simplicial complex with an ordering $\leqslant$ on the vertices of K. The complex $C_\bullet(K;R)$ has p-th graded piece defined as the free R-module generated by the oriented p-simplexes $[a_0,\dots,a_p]$ with orientation induced by $\leqslant$. When $R=\mathbb{Z}$, we obtain $C_p(K)$.
The universal property of free modules implies that

$$\partial_p([a_0,\dots,a_p]) \quad := \quad \sum_{i=0}^{p}[a_0,\dots,\widehat{a_i},\dots,a_p] \in C_{p-1}(K;R)$$

defines an R-module homomorphism $\partial_p : C_p(K;R) \longrightarrow C_{p-1}(K;R)$. Also $\partial_p\partial_{p-1}=0$ so $C_\bullet(K;R)$ with differentials ∂_p is a complex of R-modules.

Definition 20.1. *The p-th homology of K with coefficients in R is $H_p(K;R) := H_p(C_\bullet(K;R))$. The p-cycles and p-boundaries are similarly defined.*

Example 20.2. (Homology with coefficients in $\mathbb{Q}$.) Applying the functor $(-)_\mathbb{Q}$ to $\partial_p : C_p(K) \longrightarrow C_{p-1}(K)$ we get a map $\partial_{p,\mathbb{Q}} : C_p(K;\mathbb{Q}) \longrightarrow C_p(K;\mathbb{Q})$ via the identification $C_p(K;\mathbb{Q}) = C_p(K)_\mathbb{Q}$. So the complexes $C_\bullet(K;\mathbb{Q})$ and $C_p(K)_\mathbb{Q}$ are the same.

Corollary 20.3. *We have an isomorphism in homology $H_p(K;\mathbb{Q}) \simeq H_p(K)_\mathbb{Q} \simeq \mathbb{Q}^{b_p}$, where b_p is the p-th Betti number.*

Proof. Since $(-)_\mathbb{Q}$ is exact, it preserves homology, so

$$\begin{aligned} H_p(K;\mathbb{Q}) &= H_p(C_\bullet(K;\mathbb{Q})) \\ &= H_p(C_\bullet(K)_\mathbb{Q}) \\ &= H_p(C_\bullet(K))_\mathbb{Q}. \end{aligned}$$

Another way to see this is that $H_p(K) \simeq \mathbb{Z}^{b_p} \oplus \mathrm{Tors}$, tensoring by $\mathbb{Q}$ kills the torsion, so we get the above isomorphism in homology. □

In fact, for any field k of characteristic zero, $H_p(K;k) \simeq k^{b_p}$. To apply a similar idea to understand $H_p(K;\mathbb{Z}/n\mathbb{Z})$, we need a new change of scalars functor from $\mathbb{Z}$ to $\mathbb{Z}/n\mathbb{Z}$.

Definition 20.4. *Let $\mathfrak{a}$ be an ideal of R, define $R/\mathfrak{a}\otimes_R - : R$-Mod$\longrightarrow R/\mathfrak{a}$-Mod by*

$$R/\mathfrak{a}\otimes_R M \quad := \quad M/\mathfrak{a}M$$

for all $M \in R$-Mod, where $\mathfrak{a}M = \{\sum r_i m_i \mid r_i \in \mathfrak{a}, m_i \in M\}$, and

$$\begin{aligned} R/\mathfrak{a}\otimes_R \varphi : M/\mathfrak{a}M &\longrightarrow M'/\mathfrak{a}M' \\ m+\mathfrak{a}M &\longmapsto \varphi(m)+\mathfrak{a}M' \end{aligned}$$

for all $\varphi \in \mathrm{Hom}_R(M,M')$.

We check that $M/\mathfrak{a}M$ has a natural structure of an $R/\mathfrak{a}$-module. Firstly $M/\mathfrak{a}M$ is an R-module, and elements of $\mathfrak{a}$ act as zero, since for any $a \in \mathfrak{a}$, $a(m+\mathfrak{a}M) = \mathfrak{a}M$. This allows us to define $(r+\mathfrak{a})(m+\mathfrak{a}M) = rm+\mathfrak{a}M$, and this is independent of choices of coset representation.
By the universal property of quotients, $R/\mathfrak{a}\otimes_R \varphi$ is a well defined homomorphism of R-modules. Exercise: since $R/\mathfrak{a}$-module structure is induced from the R-module structure, it is in fact an $R/\mathfrak{a}$-module homomorphism.

Proposition 20.5. *$R/\mathfrak{a}\otimes_R -$ is a functor.*

Proof. It is clear that $R/I\otimes_R \mathrm{id} = \mathrm{id}$. Let $\varphi \in \mathrm{Hom}_R(M',M)$, and $\psi \in \mathrm{Hom}_R(M,M'')$, denote $\tilde{\varphi} = R/\mathfrak{a}\otimes_R \varphi$, we have

$$\begin{aligned} (\widetilde{\psi\circ\varphi})(m+\mathfrak{a}M) &= (\psi\varphi)(m)+\mathfrak{a}M \\ &= \psi(\varphi(m)+\mathfrak{a}M) \\ &= (\tilde{\psi}\circ\tilde{\varphi})(m+\mathfrak{a}M). \end{aligned}$$

□

21. Properties of $R/\mathfrak{a} \otimes_R -$

Let R and S be rings and $M, N \in R$-Mod. Let $\varphi, \psi \in \mathrm{Hom}_R(M, N)$. Define

$$\begin{aligned} \varphi + \psi : M &\longrightarrow N \\ m &\longmapsto \varphi(m) + \psi(m) \end{aligned}$$

This is an R-module homomorphism, since it's a group homomorphism and for any $r \in R$, $m \in M$,

$$\begin{aligned} (\varphi + \psi)(rm) &= \varphi(rm) + \psi(rm) \\ &= r(\varphi(m) + \psi(m)) \\ &= r(\varphi + \psi)(m). \end{aligned}$$

Proposition 21.1.

(1) With addition defined as above, $\mathrm{Hom}_R(M, N)$ *is an abelian group.*
(2) If R *is commutative then* $\mathrm{Hom}_R(M, N)$ *is an* R*-module with scalar multiplication,* $(r\varphi)(m) = r\varphi(m)$ *where* $\varphi \in \mathrm{Hom}_R(M, N)$, $r \in R$ *and* $m \in M$.

Proposition 21.2. *If* R *is commutative, then we have the following isomorphism of* R*-modules*

(1) $\mathrm{Hom}_R(R, M) \longrightarrow M$ *given by* $\varphi \longmapsto \varphi(1)$
(2) $M \times \ldots \times M \longrightarrow \mathrm{Hom}_R(R \oplus \ldots \oplus R, M)$ *(n summands on both sides) given by* $A := (a_1, \ldots, a_n) \longmapsto \varphi_A$ *where* $\varphi_A(r_1, \ldots, r_n) = \sum_{i=1}^n r_i a_i$.
(3) $\mathrm{Hom}_R(R^n, R^m) \longrightarrow R^{m \times n}$.

Assertion 2 follows from 1 by universal property of direct sums, and 3 follows from 2 when $M = R^m$.

Example 21.3. Let $R = \mathbb{Z}$ reacall tat $\mathbb{Z}$-Mod$= \mathfrak{Ab}$. We saw a group homomorphism $\mathbb{Z}^n \longrightarrow \mathbb{Z}^m$ are given by matrix multiplication by some $m \times n$-matrix with entries in $\mathbb{Z}$.

Definition 21.4. *Let* $F : R$*-Mod*$\longrightarrow S$*-Mod be a functor. We say that* F *is* ***additive*** *if the induced map on Hom sets is a group homomorphism.*

Additive functors take complexes to complexes, since $0 = F(0) = F(\partial_p \partial_{p-1}) = F(\partial_p)F(\partial_{p-1})$.

Proposition 21.5.

(1) $R/\mathfrak{a} \otimes_R -$ *is additive.*
(2) Let $M_i \in R$*-Mod,* $i \in I$*, then*

$$R/\mathfrak{a} \otimes_R \left(\bigoplus_{i \in I} M_i \right) \simeq \bigoplus_{i \in I} R/\mathfrak{a} \otimes_R M.$$

Proof. Let $F = R/\mathfrak{a} \otimes_R -$ and $\varphi, \psi \in \mathrm{Hom}_R(M, N)$,

$$\begin{aligned} (F(\varphi) + F(\psi))(m + \mathfrak{a}M) &= F(\varphi)(m + \mathfrak{a}M) + F(\psi)(m + \mathfrak{a}M) \\ &= \varphi(m) + \mathfrak{a}N + \psi(m) + \mathfrak{a}N \\ &= (\varphi + \psi)(m) + \mathfrak{a}N \\ &= F(\varphi + \psi)(m + \mathfrak{a}M) \end{aligned}$$

so F is additive. Note that we have morphisms $\alpha_j : M_j \longmapsto \bigoplus_{i \in I} M_i$, so $F(\alpha_j) : F(M_j) \longrightarrow F\left(\bigoplus_{i \in I} M_i\right)$. By the universal property of direct sums, we have a morphism,

$$\bigoplus_{i \in I} F(M_i) \longrightarrow F\left(\bigoplus_{i \in I} M_i \right).$$

One can show that the map is $(m_i + \mathfrak{a}M_i) \longmapsto (m_i) + \mathfrak{a}(\bigoplus M_i)$. This is clearly surjective, and injectivity follows from $\mathfrak{a}\left(\bigoplus M_i\right) = \bigoplus \mathfrak{a}M_i$. $\square$

Example 21.6. First note that $R/\mathfrak{a} \otimes_R R = R/\mathfrak{a}$. Let K be a simplicial complex, then

$$\begin{aligned} C_p(K; \mathbb{Z}/n\mathbb{Z}) &= \bigoplus_{[a_0, \ldots, a_p]} \mathbb{Z}/n\mathbb{Z}[a_0, \ldots, a_p] \\ &\simeq \bigoplus_{[a_0, \ldots, a_p]} \mathbb{Z}/n\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}[a_0, \ldots, a_p] \\ &\simeq \mathbb{Z}/n\mathbb{Z} \otimes_{\mathbb{Z}} C_p(K) \end{aligned}$$

In fact, $C_\bullet(K; \mathbb{Z}/n\mathbb{Z}) \simeq \mathbb{Z}/n\mathbb{Z} \otimes_{\mathbb{Z}} C_\bullet(K)$.

Definition 21.7. *A functor* $F : R$*-Mod*$\longrightarrow S$*-Mod is* ***right exact*** *if it takes any exact sequence of the form* $M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ *in* R*-Mod to an exact sequence* $F(M') \longrightarrow F(M) \longrightarrow F(M'') \longrightarrow 0$ *in* S*-Mod.*

Proposition 21.8. *The functor* $R/\mathfrak{a} \otimes_R -$ *is right exact.*

Proof. Let $M' \xrightarrow{\varphi} M \xrightarrow{\psi} M'' \longrightarrow 0$ be an exact sequence in R-Mod. Now $(F\psi)(m + \mathfrak{a}M) = \psi(m) + \mathfrak{a}M''$ so if ψ is surjective, $F\psi$ is also surjective. It remains to check $\ker(F\psi) \subseteq \operatorname{im}(F\varphi)$. Let $m + \mathfrak{a}M \in \ker(F\psi)$, so $\psi(m) \in \mathfrak{a}M''$. Let $\psi(m) = \sum r_i m_i''$ with $r_i \in \mathfrak{a}$ and $m_i'' \in M''$. Since ψ is surjective, $m_i'' = \psi(m_i)$ for some $m_i \in M$. Therefore $\psi(m) = \sum r_i \psi(m_i) = \psi\left(\sum r_i m_i\right)$. Exactness at M implies $m - \sum r_i m_i \in \operatorname{im}(\varphi)$, so let $m - \sum r_i m_i = \varphi(m')$ for some $m' \in M'$. Hence $n + \mathfrak{a}M = \varphi(m') + \mathfrak{a}M \in \operatorname{im}(F\varphi)$, hence we have exactness at $F(M)$. □

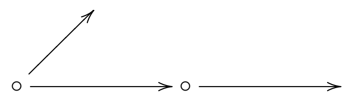
Example 21.9. Let $R = \mathbb{Z}$ consider the exact sequence $0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}/m\mathbb{Z} \longrightarrow 0$. Applying the functor $\mathbb{Z}/n\mathbb{Z} \otimes_{\mathbb{Z}} -$, we get

$$\mathbb{Z}/n\mathbb{Z} \longrightarrow \mathbb{Z}/n\mathbb{Z} \longrightarrow \mathbb{Z}/n\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/m\mathbb{Z} = \mathbb{Z}/(m\mathbb{Z} + n\mathbb{Z}) \longrightarrow 0$$

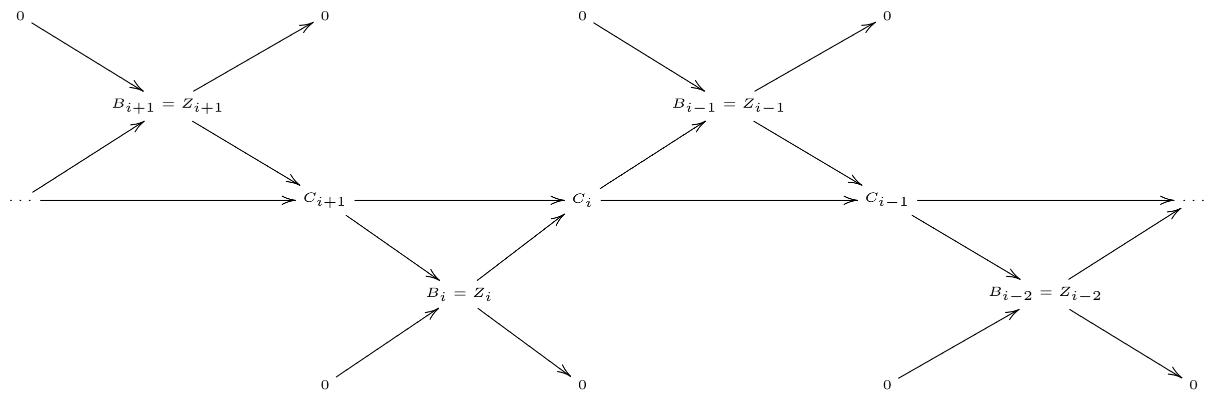
If $n = m$, then the first map is the zero map, so it is not injective.

22. Free resolutions

Consider an exact complex $C_\bullet$ in R-Mod. We can split $C_\bullet$ into short exact sequences as follows: the following diagram



is commutative with all rows and diagonals exact. Note that $B_i = Z_i$. This gives a series of short exact sequences



Conversely, given short exact sequences of the form above satisfying $B_i = Z_i$, one can "splice" them together to produce a long exact sequence $C_\bullet$.

Corollary 22.1. *Let $F : R$-Mod$\longrightarrow S$-Mod is exact iff it takes short exact sequences to short exact sequences.*

22.1. Free resolutions.

Definition 22.2. *Let $M \in R$-Mod. A **free resolution** of M is an exact complex $F_\bullet \longrightarrow M \longrightarrow 0$ such that each F_i is a free R-module. The resolution is finite if every F_i is finitely generated.*

Example 22.3. Let $R = \mathbb{Z}$ and $M = (\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z}) \oplus \mathbb{Z}$ has a finite free resolution

$$\mathbb{Z}^2 \longrightarrow \mathbb{Z}^3 \longrightarrow M \longrightarrow 0$$

Let $M \in R$-Mod and S be a subset of M. The **submodule generated** by S is the unique smallest submodule containing S. It exists and is

$$RS \quad = \quad \left\{\sum r_i m_i \mid r_i \in R, m_i \in S\right\}.$$

Theorem 22.4. *Any $M \in R$-Mod has a free resolution.*

Proof. We have an exact sequence $0 \longrightarrow K \longrightarrow \bigoplus_{m \in M} Rm \xrightarrow{\pi} M \longrightarrow 0$. Repeating this, and glueing the resulting short exact sequences gives a free resolution of M. □

Theorem 22.5. *Consider two complexes in R-Mod, $\tilde{F}_\bullet : F_\bullet \longrightarrow M \longrightarrow 0$ and $\tilde{C}_\bullet : C_\bullet \longrightarrow M' \longrightarrow 0$ wih $F_\bullet$ a complex where F_i are free and $\tilde{C}_\bullet$ is exact. Let $\varphi \in \mathrm{Hom}_R(M, M')$.*

(1) There exists a chain map $\tilde{\varphi}_\bullet : \tilde{F}_\bullet \longrightarrow \tilde{C}_\bullet$ such that $\tilde{\varphi}_{-1} = \varphi$.
(2) Given two such $\tilde{\varphi}_\bullet, \tilde{\varphi}'_\bullet : \tilde{F}_\bullet \longrightarrow \tilde{C}_\bullet$, the induced chain maps $\varphi_\bullet, \varphi'_\bullet : F_\bullet \longrightarrow C_\bullet$ are chain homotopic.
(3) If $M = M'$ and $\varphi = \mathrm{id}$, then any two free resolutions $F_\bullet \longrightarrow M \longrightarrow 0$, $C_\bullet \longrightarrow M \longrightarrow 0$ are such that $F_\bullet \longrightarrow 0$ and $C_\bullet \longrightarrow 0$ are chain equivalent.

Lemma 22.6. *Let $\varphi : F \longrightarrow M$ be an R-module morphism from a free module to M, then there exists an element $\varphi \in \mathrm{Hom}_R(F, N)$ such that*

$$\begin{array}{ccc} & & F \\ & \tilde{\varphi} \swarrow & \downarrow \varphi \\ N & \longrightarrow & M \longrightarrow 0 \end{array}$$

commutes.

Proof. Let $\{e_i\}_{i \in I}$ be a basis for F. Define $\tilde{\varphi}$ by sending e_i to any preimage of $\varphi(e_i)$. □

Proof. (of theorem) Define chain map $\tilde{\varphi}_\bullet$ by induction on degree.

$$\begin{array}{ccccc} & & F_0 & & \\ & & \downarrow & & \\ & & M & & \\ & & \downarrow & & \\ C_0 & \longrightarrow & M' & \longrightarrow & 0 \end{array}$$

so there exists $\tilde{\varphi}_0 : F_0 \longrightarrow C_0$ by lemma . The inductive step is similar to the case for $i = 1$

$$\begin{array}{cccccccc} \partial_0 \tilde{\varphi}_0 \partial_1 : & F_1 & \xrightarrow{\partial_0} & F_0 & \longrightarrow & C_0 & \longrightarrow & M' \\ || & & & & & & & \\ \varphi \partial_0 \partial_1 = 0 & & & & & & & \end{array}$$

so $\mathrm{im}(F_1 \longrightarrow F_0 \longrightarrow C_0) \subseteq \ker(C_0 \longrightarrow M') = \mathrm{im}(C_1 \longrightarrow C_0)$. Apply lemma to

$$\begin{array}{ccc} & & F \\ & & \downarrow \\ C_1 & \longrightarrow & \mathrm{im}(C_1 \longrightarrow C_0) \longrightarrow 0 \end{array}$$

to define $\tilde{\varphi}_1$ etc.
By the previous parts there exist chain maps $\tilde{\varphi}_\bullet : \tilde{F}_\bullet \longrightarrow \tilde{C}_\bullet$ and $\tilde{\psi}_\bullet : \tilde{C}_\bullet \longrightarrow \tilde{F}_\bullet$. The maps $\tilde{\psi}_\bullet \circ \tilde{\varphi}_\bullet$ and $\mathrm{id}_\bullet$ are homotopic by part 2. □

23. Derived functors

Let R, S be rings and $F : R\text{-Mod} \longrightarrow S\text{-Mod}$ be a functor.

Lemma 23.1. *If F is right exact then F is additive and for $M, M' \in R$-Mod,*

$$F(M \oplus M') \simeq F(M) \oplus F(M') \tag{4}$$

Proof is omitted since in most cases, additivity and (4) can be checked directly.

Lemma 23.2. *If F is additive, then it preserves homotopies.*

Proof. Consider two chain maps $\varphi_\bullet$ and $\psi_\bullet$, and chain homotopy $\{s_p : C_p \longrightarrow C'_{p-1}\}$, so that

$$\varphi_p - \psi_p = s_{p-1}\partial_p + \partial_{p+1}s_p.$$

Since F is functorial and additive, $F(\varphi_p) - F(\psi_p) = F(s_{p-1})F(\partial_p) + F(\partial_{p+1})F(s_p)$, that is $F(s_\bullet)$ is a chain homotopy between $F(\varphi_\bullet)$ and $F(\psi_\bullet)$. □

Assume the the following that F is right exact, we seek to construct functors $L_iF : R\text{-Mod} \longrightarrow S\text{-Mod}$ for all $i > 0$, such that for all exact sequences $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ in R-Mod, we obtain a long exact sequence

$$\ldots \longrightarrow L_1F(M') \longrightarrow L_1F(M) \longrightarrow L_1F(M'') \longrightarrow F(M') \longrightarrow F(M) \longrightarrow F(M'') \longrightarrow 0$$

in S-Mod.
Let $M \in R$-Mod, pick a free resolution $\tilde{P}_\bullet : P_\bullet \longrightarrow M \longrightarrow 0$, F additive implies $F(P_\bullet)$ is a complex in S-Mod. Define $L_iF(M) := H_i(F(P_\bullet))$. Check well defined, that is, independent of the choice of resolution. Consider another free resolution $P'_\bullet \longrightarrow M \longrightarrow 0$. Uniqueness up to homotopy says that there are chain

maps $\varphi_\bullet : P_\bullet \longrightarrow P'_\bullet$ unique up to homotopy and chain homotopy inverse $\psi_\bullet : P'_\bullet \longrightarrow P_\bullet$ with commutative diagram

$$\begin{array}{ccc} P_\bullet & \longrightarrow & M \\ \varphi_\bullet \downarrow & & \downarrow \text{id} \\ P'_\bullet & \longrightarrow & M \end{array}$$

$$\begin{array}{ccc} P_\bullet & \longrightarrow & M \\ \psi_\bullet \uparrow & & \uparrow \text{id} \\ P'_\bullet & \longrightarrow & M \end{array}$$

Lemma 23.2 implies that there are canonically chosen homomorphisms

$$H_i(F\varphi_\bullet) : H_i(FP_\bullet) \longrightarrow H_i(FP'_\bullet)$$

and

$$H_i(F\psi_\bullet) : H_i(FP'_\bullet) \longrightarrow H_i(FP_\bullet).$$

Lemma 23.2 also imiplies that $F(\varphi_\bullet \circ \psi_\bullet) = F(\varphi_\bullet) \circ F(\psi_\bullet)$ and $(F\psi_\bullet) \circ (F\varphi_\bullet)$ are homotopic to id, $F\varphi_\bullet$ and $F\psi_\bullet$ are chain homotopy inverses. Therefore $H_i(F\varphi_\bullet)$ and $H_i(F\psi_\bullet)$ are canonically chosen isomorphisms. So we can assume it is well defined using the trick in lecture 12 by picking any resolutions.

Proposition 23.3.
(1) $L_0F = F$
(2) If F *is exact, then* L_iF *is the zero functor for all* $i > 0$.

Proof. Consider a free resolution $P_1 \longrightarrow P_0 \xrightarrow{\partial_0} M \longrightarrow 0$, right exactness of F means that $M \simeq FP_0/\ker(F\partial_0) = FP_0/\operatorname{im}(F\partial_1) = L_0F(M)$. For part 2, choose a resolution $0 \longrightarrow F \longrightarrow F \longrightarrow 0$ and compute homology. □

Let $M, M' \in R$-Mod and $\varphi \in \operatorname{Hom}_R(M, M')$. Consider free resolutions $P_\bullet \longrightarrow M \longrightarrow 0$ and $P'_\bullet \longrightarrow M' \longrightarrow 0$ of M and M'. There exists a chain map $\varphi_\bullet : P_\bullet \longrightarrow P'_\bullet$ so the diagram

$$\begin{array}{ccc} P_\bullet & \longrightarrow & M \longrightarrow 0 \\ \varphi_\bullet \downarrow & & \downarrow \varphi \\ P'_\bullet & \longrightarrow & M' \longrightarrow 0 \end{array}$$

commutes. Now define

$$L_iF(\varphi) : L_iF(M) \longrightarrow L_iF(M')$$

by $L_iF = H_i(F\varphi_\bullet)$, which is well defined by using the previous arguments and lemma 23.2. We get

Proposition 23.4. $L_iF : R$*-Mod*$\longrightarrow S$*-Mod is a functor.*

Theorem 23.5. *Let* $F : R$*-Mod*$\longrightarrow S$*-Mod be a right exact functor. For any exact sequence* $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ *in R-Mod, there exists an long exact sequence*

$$\ldots \longrightarrow L_1F(M') \longrightarrow L_1F(M) \longrightarrow L_1F(M'') \longrightarrow F(M') \longrightarrow F(M) \longrightarrow F(M'') \longrightarrow 0$$

Proof. (Sketch) The horseshoe lemma implies we have free resolutions $P'_\bullet \longrightarrow M' \longrightarrow 0$ and $P''_\bullet \longrightarrow M'' \longrightarrow 0$ ad $P_\bullet \longrightarrow M \longrightarrow 0$ where $P_i = P'_i \oplus P''_i$. Further there is a short exact sequence of complexes

$$0 \longrightarrow P'_\bullet \xrightarrow{\varphi_\bullet} P_\bullet \xrightarrow{\psi_\bullet} P''_\bullet \longrightarrow 0$$
$$\ldots$$

where $\varphi_p : P'_p \longrightarrow P_p$ is canonical injection and $\psi_p : P_p \longrightarrow P''_p$ is a canonical projection. Lemma 1 implies that we have a short exact sequence of complexes $0 \longrightarrow FP'_\bullet \longrightarrow FP_\bullet \longrightarrow FP''_\bullet \longrightarrow 0$ (*). The theorem follows from the long exact sequence in homology applied to (*). □

The functor L_iF is called the i-th left derived functor of F.

24. Tor

Let R be a commutative ring and $\mathfrak{a} \lhd F$, then we have a right exact functor $R/\mathfrak{a} \otimes_R - : R$-Mod$\longrightarrow R/\mathfrak{a}$-Mod. The accompanying left derived functors $L_i(R/\mathfrak{a} \otimes_R -)$ is denoted $\operatorname{Tor}_i^R(R/\mathfrak{a}, -)$. By theorem 23.5, there is a long exact sequence in Tor.

Example 24.1. Let $M = R^n$, and $0 \longrightarrow R^n$ be a free resolution for M. Then we obtain $\operatorname{Tor}_0^R(R/\mathfrak{a}, R^n) = (R/\mathfrak{a})^n$, and the higher Tor's are all zero.

Lemma 24.2.

(1) Let $M, M' \in R$-Mod, then

$$\operatorname{Tor}_i^R(R/\mathfrak{a}, M \oplus M') = \operatorname{Tor}_i^R(R/\mathfrak{a}, M) \oplus \operatorname{Tor}_i^R(R/\mathfrak{a}, M').$$

(2) The functors $\operatorname{Tor}_i^R(R/\mathfrak{a}, -)$ is additive.

Proof. We will prove 1 only. Let $P_\bullet$ and $P'_\bullet$ be free resolutions of M and M'. Then the direct sum $P_\bullet \oplus P'_\bullet$ is a free resolution for $M \oplus M'$ (exact since homology commutes with direct sums).

$$\begin{aligned}\operatorname{Tor}_i^R(R/\mathfrak{a}, M \oplus M') &= H_i(R/\mathfrak{a} \otimes_R (P_\bullet \oplus P'_\bullet)) \\ &= H_i((R/\mathfrak{a} \otimes_R P_\bullet) \oplus (R/\mathfrak{a} \otimes_R P'_\bullet)) \\ &= H_i(R/\mathfrak{a} \otimes_R P_\bullet) \oplus H_i(R/\mathfrak{a} \otimes_R P'_\bullet) \\ &= \operatorname{Tor}_i^R(R/\mathfrak{a}, M) \oplus \operatorname{Tor}_i^R(R/\mathfrak{a}, M').\end{aligned}$$

□

Note 24.3. Let $x \in R$, then since R is commutative, we have a module morphism $M \xrightarrow{\cdot x} M$. The image is $xM = (x)M$.

Proposition 24.4. *Let x be a non zero divisor. Then for $M \in R$-Mod, we have*

$$\operatorname{Tor}_i^R(R/(x), M) = \begin{cases} M/xM & \text{if } i = 0 \\ 0 & \text{if } i > 1 \\ \ker(M \xrightarrow{\cdot x} M) & \text{if } i = 1 \end{cases}.$$

Proof. Let $P_\bullet$ be a free resolution of M. We have an exact sequence of complexes

$$0 \longrightarrow P_\bullet \xrightarrow{\cdot x} P_\bullet \longrightarrow P_\bullet/(xP_\bullet) \longrightarrow 0$$

The long exact sequence in cohomology gives

$$\ldots \longrightarrow H_i(P_\bullet) \longrightarrow H_i(P_\bullet/(xP_\bullet)) \longrightarrow H_{i-1}(P_\bullet) \longrightarrow \ldots$$

so for $i > 1$, $H_i(P_\bullet/(xP_\bullet)) = \operatorname{Tor}_i^R(R/(x), M)$ vanishes. For $i = 1$, we get

$$0 \longrightarrow H_1(R/(x) \otimes_R P_\bullet) \longrightarrow M \xrightarrow{\cdot x} M$$

so $\operatorname{Tor}_1^R(R/(x), M) = \ker(M \xrightarrow{\cdot x} M)$. □

Example 24.5. Let $R = \mathbb{Z}$, and $\mathfrak{a} = 6\mathbb{Z}$ compute $\operatorname{Tor}_i^{\mathbb{Z}}(\mathbb{Z}/6\mathbb{Z}, \mathbb{Z}/9\mathbb{Z})$. Using the above proposition, we get

$$\operatorname{Tor}_i^R(\mathbb{Z}/6\mathbb{Z}, M) = \begin{cases} \frac{\mathbb{Z}/9\mathbb{Z}}{6(\mathbb{Z}/9\mathbb{Z})} = \frac{\mathbb{Z}}{6\mathbb{Z}+9\mathbb{Z}} = \mathbb{Z}/3\mathbb{Z} & \text{if } i = 0 \\ 0 & \text{if } i > 1 \\ \operatorname{Ann}_{\mathbb{Z}}(6) & \text{if } i = 1 \end{cases}.$$

$$\begin{aligned}\operatorname{Ann}_{\mathbb{Z}}(6) &= \ker(\mathbb{Z}/9\mathbb{Z} \xrightarrow{\cdot 6} \mathbb{Z}/9\mathbb{Z}) \\ &= \ker(\mathbb{Z}/9\mathbb{Z} \xrightarrow{\cdot 3} \mathbb{Z}/9\mathbb{Z}) \\ &= 3\mathbb{Z}/9\mathbb{Z} \\ &\simeq \mathbb{Z}/3\mathbb{Z}\end{aligned}$$

Example 24.6. Let $R = k[x, y]$, $\mathfrak{a} = (x, y)$. We compute $\operatorname{Tor}_i^R(R/\mathfrak{a}, k)$. Consider the Koszul resolution

$$0 \longrightarrow R \xrightarrow{\begin{pmatrix} y \\ -x \end{pmatrix}} R^2 \xrightarrow{\begin{pmatrix} x & y \end{pmatrix}} R \longrightarrow k \longrightarrow 0$$

of k. Then both maps become the zero maps after tensoring by $R/\mathfrak{a}$, so we obtain

$$\operatorname{Tor}_i^R(R/\mathfrak{a}, k) = \begin{cases} k^2 & \text{if } i = 1 \\ k & \text{if } i = 0, 2 \\ 0 & \text{otherwise} \end{cases}.$$

25. Universal coefficient theorem. Graded Algebras

Recall $\mathfrak{TrTop}$ = category of triangulable topological spaces and continuous maps and $\mathfrak{Tr}H$ =category of triangulable topological spaces and homotopy equivalence clases of continuous maps.

Theorem 25.1. *There exists a functor $H_p(-; \mathbb{Z}/n\mathbb{Z}) : \mathfrak{Tr}H \longrightarrow \mathbb{Z}/n\mathbb{Z}$-Mod, and hence a functor $H_p(-; \mathbb{Z}/n\mathbb{Z}) : \mathfrak{TrTop} \longrightarrow \mathfrak{Ab}$. A similar statement holds for $\tilde{H}_p$,*

Proof. As for coefficient $\mathbb{Z}$ case. For simplicial complexes K, L we can define $H_p(|K|; \mathbb{Z}/n\mathbb{Z}) = H_p(K; \mathbb{Z}/n\mathbb{Z})$. Also for continuous $h : |K| \longrightarrow |L|$ there is a simplicial approximation $f : \operatorname{sd}^N K \longrightarrow L$ for h. Then we can define $H_p(h; \mathbb{Z}/n\mathbb{Z}) : H_p(|\operatorname{sd}^N K|; \mathbb{Z}/n\mathbb{Z}) \longrightarrow H_p(|L|; \mathbb{Z}/n\mathbb{Z})$. □

25.1. **Universal coefficient theorem.**

Theorem 25.2. *Let K be a simplicial complex. Then there exists a split exact sequence in $\mathbb{Z}/n\mathbb{Z}$-Mod*

$$0 \longrightarrow \mathbb{Z}/n\mathbb{Z} \otimes_{\mathbb{Z}} H_p(K) \longrightarrow H_p(K;\mathbb{Z}/n\mathbb{Z}) \longrightarrow \operatorname{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, H_{p-1}(K)) \longrightarrow 0$$

for each p.

Proof. For each p, there exists an exact sequence

$$0 \longrightarrow C_p(K) \xrightarrow{\cdot n} C_p(K) \longrightarrow C_p(K)/nC_p(K) \longrightarrow 0$$

We know that $C_p(K)/nC_p(K) \simeq \mathbb{Z}/n\mathbb{Z} \otimes_{\mathbb{Z}} C_p(K) \simeq C_p(K;\mathbb{Z}/n\mathbb{Z})$. Hence we have a short exact sequence of complexes

$$0 \longrightarrow C_\bullet(K) \longrightarrow C_\bullet(K) \longrightarrow C_\bullet(K,\mathbb{Z}/n\mathbb{Z}) \longrightarrow 0$$

But $H_p(K,\mathbb{Z}/n\mathbb{Z}) = H_p(C_\bullet(K;\mathbb{Z}/n\mathbb{Z}))$, so we get a long exact sequence

$$\ldots \longrightarrow H_p(K) \xrightarrow{\cdot n} H_p(K) \longrightarrow H_p(K,\mathbb{Z}/n\mathbb{Z}) \longrightarrow H_{p-1}(K) \longrightarrow \ldots$$

From this we can extract a short exact sequence

$$0 \longrightarrow \frac{H_p(K)}{nH_p(K)} \longrightarrow H_p(K,\mathbb{Z}/n\mathbb{Z}) \longrightarrow \ker(H_{p-1}(K) \xrightarrow{n} H_{p-1}(K)) \longrightarrow 0$$

but $\frac{H_p(K)}{nH_p(K)} \simeq \mathbb{Z}/n\mathbb{Z} \otimes_{\mathbb{Z}} H_p(K)$ and $\ker(H_{p-1}(K) \xrightarrow{n} H_{p-1}(K)) \simeq \operatorname{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}; H_{p-1}(K))$ by proposition 24.4, so the theorem follows. □

Example 25.3. Let $P = \mathbb{P}^2_{\mathbb{R}}$.
For $p = 0$ we have $0 \longrightarrow \mathbb{Z}/2\mathbb{Z} \simeq \mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} H_0(P) \longrightarrow H_0(P;\mathbb{Z}/2\mathbb{Z}) \longrightarrow \operatorname{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z};\mathbb{Z}) = 0$.
For $p = 1$ we have $0 \longrightarrow \mathbb{Z}/2\mathbb{Z} \simeq \mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} H_1(P) \longrightarrow H_1(P;\mathbb{Z}/2\mathbb{Z}) \longrightarrow \operatorname{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z};\mathbb{Z}) = 0$.
For $p = 2$ we have $0 \longrightarrow \mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} 0 \longrightarrow H_2(P;\mathbb{Z}/2\mathbb{Z}) \longrightarrow \operatorname{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z};\mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$.
So

$$H_p(P;\mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$$

for all $p = 0, 1, 2$. Also $\mathbb{Z}/2\mathbb{Z}$ is a field, so splitting of the universal coefficient exact sequence is automatic.

Remark 25.4. For an n-dimensional compact triangulable manifold X, then $H_n(X;\mathbb{Z}/2\mathbb{Z}) \simeq \mathbb{Z}/2\mathbb{Z}$. If X is orientable, then $H_n(X) \simeq \mathbb{Z}$ otherwise $H_n(X) = 0$.

25.2. **Graded algebras.** Assumee that k is a field.

Definition 25.5. *An $\mathbb{N}$* ***-graded k -algebra*** *is a ring A with an abelian group decomposition $A = \bigoplus_{n\geqslant 0} A_n$ such that*

1. *$k \subseteq A_0$ as a subring and above is a vector space decomposition.*
2. *$\alpha a = a\alpha$ for any $a \in A$ and $\alpha \in k$*
3. *$A_iA_j \subseteq A_i$ for all $i, j \in \mathbb{N}$.*

The nonzero elements of A_i are called the ***homogeneous elements*** *of* ***degree*** *i.*

Example 25.6. $A = k[x_1, \ldots, x_n]$ is a graded k-algebra.

Proposition 25.7. *Let A be a graded k-algebra. An ideal $\mathfrak{a} \lhd A$ is called* ***graded*** *or* ***homogeneous*** *if $\mathfrak{a} = \bigoplus_{n\geqslant 0} \mathfrak{a}_n$ where $\mathfrak{a}_n = \mathfrak{a} \cap A_n$. In this case, we get a graded k-algebra $A/\mathfrak{a} = \bigoplus_{n\geqslant 0} A_n/\mathfrak{a}_n$.*

Example 25.8. Any ideal generated by homogeneous elements is homogeneous, for instance $\mathfrak{a} = (x, y^2)$ in $k[x, y]$.

Definition 25.9. *Let A be a graded k-algebra. Then $M \in A$-Mod is a* ***graded module*** *if it comes with a vector space decomposition $M = \bigoplus_{n\in\mathbb{Z}} M_n$ satisfying the following $A_iM_j \subseteq M_{i+j}$.*

Example 25.10. A is a graded A-module, ($A_j = 0$ for $j < 0$).

Let M be a graded A-module, for $l \in \mathbb{Z}$ define $M[l]$ to be the shift, $M[l] = M_l$.

26. The category of graded modules

Definition 26.1. *An A-module morphism is* ***graded*** *if it preserves degree.*

Example 26.2. Let $A = k[x]$, the only graded homomorphism $A \longrightarrow A[-1]$ is the zero map. The only graded homomorphism $A[-1] \longrightarrow A$ is multiplication by αx for $\alpha \in k$.

Proposition 26.3. *The class of graded A-modules and graded homomorpihsms forms a category denoted A-Gr. The set of graded homomorpihsms from M to N is denoted $\operatorname{GrHom}_A(M, N)$ and this is a subspace of both $\operatorname{Hom}_A(M, N)$ and $\operatorname{Hom}_k(M, N)$.*

26.1. **Graded submodules and quotients.**

Proposition 26.4.

(1) *An A-submodule M' of M is* **graded** *if $M' = \bigoplus_{n\in\mathbb{Z}} M'_j$ where $M'_j := M' \cap M_j$. In this case M' is a graded A-module, as is the quotient M/M'.*
(2) *The kernels and images of graded homomorphisms and are graded A-modules.*
(3) *For $M, N \in A$-Gr, $M \oplus N = \bigoplus(M_j \oplus N_j)$.*

26.2. **Graded free resolutions.** The propositions above imply that the notion of complexes and homology in A are well defined, and both are graded A-modules.

Definition 26.5. *A* ***graded free*** *A-module is one of the form $\bigoplus_{i\in I} A[d_i]$ for some $d_i \in \mathbb{Z}$. A* ***graded free resolution*** *of $M \in A$-Gr is an exact sequence in A-Gr of the form*

$$\ldots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

where each P_j is graded free.

Note 26.6. The graded homomorphism $\bigoplus_{i=1}^n A[d_i] \longrightarrow \bigoplus_{j=1}^m A[e_j]$ given by an $m \times n$ -matrix of homogeneous elements in A with (i,j)-th entry in $A_{e_i - d_j}$.

Example 26.7. Continuing example 24.6, let $A = k[x,y]$, $k = A/(x,y)$, we have a graded free resolution

$$0 \longrightarrow A[-2] \xrightarrow{\begin{pmatrix} y \\ -x \end{pmatrix}} A[-1]^2 \xrightarrow{\begin{pmatrix} x & y \end{pmatrix}} A \longrightarrow k \longrightarrow 0.$$

26.3. **Hilbert basis theorem.**

Theorem 26.8. *Let $A = k[x_1,\ldots,x_n]/\mathfrak{a}$ where $\mathfrak{a}$ is some ideal of $k[x_1,\ldots,x_n]$. If M is a finitely generated A-module, then every submodule of M is finitely generated. In particular, M has a free resolution $P_\bullet \longrightarrow M \longrightarrow 0$ with every P_j finitely generated.*

If $\mathfrak{a}$ is homogeneous and M is graded then M has a graded free resolution $P_\bullet \longrightarrow M \longrightarrow 0$ with each P_j a graded free. Let B be another graded k-algebra. A **graded homomorphism of algebras** $\varphi : A \longrightarrow B$ is a ring homomorphism satisfying

(1) φ restricts to the identity on $k \subseteq A_0$.
(2) $\varphi(A_j) \subseteq B_j$.

Example 26.9. Let $\mathfrak{a}$ be a homogeneous ideal of A, the quotient map $A \longrightarrow A/\mathfrak{a}$ is a homomorphism of graded k algebras.

Proposition 26.10. *There is a restriction of scalars functor $\operatorname{res}_A^B : B$-Gr$\longrightarrow A$-Gr defined by $\operatorname{res}_A^B(M) :=$ M as an abelian group and the same vector space decomposition, but the scalar multiplication by $a \in A$ is defined as scalar multiplication $\varphi(a) \in B$. Note that we often write M_A for $\operatorname{res}_A^B M$.*

Proof. Let $f : M \longrightarrow N$ be a graded B-module homomorphism, then $M_A \xrightarrow{f} N_A$ is automatically a graded A-module homomorphism.

□

26.4. **Grading on Tor.** Let $\mathfrak{a}$ be a homogeneous ideal of A.

Lemma 26.11. *Let $M \in A$-Gr, $\mathfrak{a}M$ is also graded.*

Proof. It suffices to show $\mathfrak{a}M = \bigoplus M_j \cap \mathfrak{a}M$. One inclusion is clear, for the other, we have

$$\mathfrak{a}M \supseteq \bigoplus_j M \cap \mathfrak{a}M \supseteq \sum_j \sum_{j_1+j_2=j} \mathfrak{a}_{j_1} M_{j_2}.$$

The two end terms are equal, so we have the lemma. □

Corollary 26.12. *We have functors $A/\mathfrak{a} \oplus_A - : A$-Gr$\longrightarrow A/\mathfrak{a}$-Gr, and $\operatorname{Tor}_i^A(A/\mathfrak{a}, -) : A$-Gr$\longrightarrow A/\mathfrak{a}$-Gr.*

Proof. Omitted □

Example 26.13. We check this for example 26.7. Apply $k\otimes_{A-}$ to the resolution before

$$0 \longrightarrow k[-2] \xrightarrow{\begin{pmatrix} y \\ -x \end{pmatrix}} k[-1]^2 \xrightarrow{\begin{pmatrix} x & y \end{pmatrix}} k \longrightarrow k \longrightarrow 0.$$

so

$$\operatorname{Tor}_i^A(k,k) = \begin{cases} 0 & i > 2 \\ k & i = 0 \\ k[-1]^2 & i = 1 \\ k[-2] & i = 2 \end{cases}$$

27. The Hilbert polynomial

Let A be a graded k-algebra. We say that $M \in A$-Gr is **locally finite** if $\dim_k M_j < \infty$ for all $j \in \mathbb{Z}$. In this case, we can define the **Hilbert polynomial** of M to be

$$h_M(j) = \dim_k(M_j)$$

Example 27.1. Let $A = M = k[x_1, \dots, x_n]$, then $h_M(j) = \binom{j+d-1}{d-1}$ if $j \geqslant 0$ and zero otherwise.

Proof. The number of monomials of the form $x_1^{j_1} \dots x_n^{j_n}$ where $j_1 + \dots + j_n = 1$ is the same as the binary digits of the form $\underbrace{0\dots0}_{j_1}1\underbrace{0\dots0}_{j_2}1\dots1\underbrace{0\dots0}_{j_n}$, and there are $\binom{j+d-1}{d-1}$ of these. □

Note that

$$\binom{j+d-1}{d-1} = \frac{(j+d-1)\dots(j+1)}{(d-1)!}$$

so the Hilbert polynomial has degree $d-1$ with rational coefficients.

Example 27.2. Let $M \in A$-Gr be locally finite, then

$$\begin{aligned} h_{M[l]}(j) &= \dim_k M_{l+j} \\ &= h_M(l+j) \end{aligned}$$

Proposition 27.3.

(1) *Consider an exact sequence in A-Gr of locally finite modules,* $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$, *then* $h_M(j) = h_{M'}(j) + h_{M''}(j)$.

(2) *Given an exact sequence of locally finite graded modules*

$$0 \longrightarrow M_m \longrightarrow \dots \longrightarrow M_n \longrightarrow 0$$

then

$$\sum_{i=m}^{n} (-1)^i h_{M_i}(j) = 0$$

Proof. Since exact sequence is graded we have an exact sequence in k-Mod, $0 \longrightarrow M'_j \longrightarrow M_j \xrightarrow{\varphi} M''_j \longrightarrow 0$, so rank-nullity on φ implies $\dim_k(M_j) = \dim_k M''_j + \dim_k M'_j$ which gives the result. This implies part 2 on splitting up the exact sequence into short exact sequences. Alternatively, consider the exact sequence in degree j, $0 \longrightarrow (M_m)_j \longrightarrow \dots \longrightarrow (M_n)_j \longrightarrow 0$ of k-vector spaces. Applying the Hopf trace formula to the identity on the exact sequence above gives proof of part 2. □

Example 27.4. Let $A = k[x, y, z]$ and $M = A/(y^2)$. We have a graded free resolution

$$0 \longrightarrow A[-2] \xrightarrow{y^2} A \longrightarrow M \longrightarrow 0$$

so we have, for $j \geqslant 2$,

$$\begin{aligned} h_M(j) &= h_A(j) - h_{A[-2]}(j) \\ &= \binom{j+2}{2} - \binom{j}{2} \\ &= 2j+1 \end{aligned}$$

This example demonstrates that having a graded free resolution of a graded module is useful for computing its Hilbert polynomial. Hilbert's Syzygy theorem tells us that we can always find a (finite!) graded free resolution for a locally finite graded A-module, when A is a polynomial ring (over k). Moreover, the minimal length of any such a resolution is equal to the number of variables in the polynomial ring.

27.1. **Hilbert Syzygy theorem.** Let $A = k[x_1, \dots, x_d]$ and $M \in A$-Gr which is locally finite such that $M_j = 0$ for $j \ll 0$. Given any exact sequence in A-Gr,

$$0 \longrightarrow P_d \longrightarrow \dots \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

with $P_{d-1}, \dots, P_0$ graded free, then P_d is graded free. In particular for any homogeneous ideal $\mathfrak{a}$ of A, we have

$$\mathrm{Tor}_i^A(A/\mathfrak{a}, M) = 0$$

for $i > d$.

Corollary 27.5. *Let $A = k[x_1, \dots, x_d]/\mathfrak{a}$ where $\mathfrak{a}$ is a homogeneous ideal and M be a finitely generated graded A-module. Then M is locally finite and for $j \gg 0$, $h_M(j)$ is a polynomial function in j with rational coefficients, called the **Hilbert polynomial** of M.*

Proof. We can restrict scalars to $k[x_1, \ldots, x_d]$ which does not change the Hilbert function. We can assume $\mathfrak{a} = 0$, note that $M_j = 0$ if j less than the smallest degree of the generators. By the Hilbert basis theorem, we have an exact sequence in A-Gr,

$$0 \longrightarrow P_d \longrightarrow \ldots \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

where $P_{d-1}, \ldots, P_0$ are finitely generated and graded free. The Syzygy's theorem and the basis theorem, P_d is graded free and finitely generated. Now P_i has form

$$P_i = \bigoplus_{j=1}^{r} A[l_j]$$

so $h_{P_i}(j)$ is a sum of $h_{A[l_i]}(j) = h_A(l_i + j)$ for $j \gg 0$, which is a polynomial. Hence h_{P_i} is a polynomial, h_M is an alternating sum of h_{P_i}'s, so is also a polynomial. □

28. Minimal resolutions and applications

Let $A = k[x_1, \ldots, x_d]/\mathfrak{a}$ with $\mathfrak{a}$ a proper homogeneous ideal. Let $\mathfrak{m}$ be the homogeneous ideal of A generated by $x_1, \ldots, x_d$, then $A_0 = k \simeq A/\mathfrak{m}$.

Proposition 28.1. *Let $M \in A$-Gr be finitely generated. Then there is a graded free resolution*

$$\ldots \longrightarrow P_2 \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\partial_0} M \longrightarrow 0$$

*where each P_i is finitely generated and every ∂_i is represented by a matrix with entries in $\mathfrak{m}$ (that is, homogeneous elements of positive degree. Such a resolution is called **minimal**.*

Example 28.2. Let $A = k[x]$ and $M = A/(x)$. We have a minimal resolution

$$0 \longrightarrow A[-1] \xrightarrow{x} A \longrightarrow M \longrightarrow 0.$$

We can take the direct sum with the exact sequence $0 \longrightarrow A \longrightarrow A \longrightarrow 0 \longrightarrow 0$, to get another graded free resolution

$$0 \longrightarrow A[-1] \oplus A \xrightarrow{T} A \oplus A \longrightarrow M \longrightarrow 0$$

where T is given by the matrix $\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}$. This is not a minimal resolution.

Theorem 28.3. *Let M be a finitely generated graded A-module and*

$$\ldots \longrightarrow P_2 \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\partial_0} M \longrightarrow 0$$

be a minimal resolution. The P_i are uniquely determined by M. More precisely, if

$$P_i = \bigoplus_{j=1}^{n_i} A[\ell_{i,j}]$$

then

$$\mathrm{Tor}_i^A(k, M) = \bigoplus_{i=1}^{n_i} k[\ell_{i,j}]$$

Proof. Applying $k \otimes_A -$ to the minimal resolution, we get

$$\ldots \longrightarrow \bigoplus_{j=1}^{n_2} k[\ell_{2,j}] \xrightarrow{k\otimes\partial_2} \bigoplus_{j=1}^{n_1} k[\ell_{1,j}] \xrightarrow{k\otimes\partial_1} \bigoplus_{j=1}^{n_0} k[\ell_{0,j}] \longrightarrow 0$$

The maps $k \otimes \partial_i$ are all zero, since ∂_i is a matrix with entries in $\mathfrak{m}$. Taking homology gives the Tor groups. Hence the number of summands, n_i, of P_i is equal to $\dim_k \mathrm{Tor}_i^A(k, M)$. Conversely, the Tor groups determine $\ell_{i,1}, \ldots, \ell_{i,n_i}$ and hence P_i. □

Example 28.4. Let $A = k[x, y, z]$ and M be a finitely generated graded A-module. Suppose w e have

$$\mathrm{Tor}_i^A(k, M) = \begin{cases} k & \text{if } i = 0 \\ k[-1] \oplus k[-2] & \text{if } i = 1 \\ k[-3] & \text{if } i = 2 \\ 0 & \text{otherwise} \end{cases}$$

We use this to find the Hilbert polynomial of M. The theorem above implies we have a minimal resolution

$$0 \longrightarrow A[-3] \longrightarrow A[-1] \oplus A[-2] \longrightarrow A \longrightarrow M \longrightarrow 0.$$

Using additivity of the Hilbert polynomial, we have for $j \gg 0$

$$\begin{aligned} h_M(j) &= h_A(j) - h_{A[-1]\oplus A[-2]}(j) + h_{A[-3]}(j) \\ &= \binom{j+2}{2} - \left(\binom{j+1}{2} + \binom{j}{2}\right) + \binom{j-1}{2} \\ &= 2. \end{aligned}$$

We can also show that M is not a submodule of A. Suppose the contrary and let a be a degree r element of M, so $(a) < M$, implying $h_{(a)}(j) \leqslant h_M(j)$ for all j. But $(a) \simeq A[-r]$, so

$$\begin{aligned} h_{(a)}(j) &= h_{A[-r]}(j) \\ &= h_A(j-c) \\ &= \binom{j-c+2}{2} \end{aligned}$$

is quadratic in j, so eventually > 2. This contradicts $h_{(a)}(j) \leqslant h_M(j)$ for all j.

Example 28.5. Let $A = k[x,y]$. Suppose M is a finitely generated with graded free resolution

$$0 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

then M has no finite dimensional (over k) graded submodules. Suppose there does exists a submodule N of M which is finite dimensional over k. Pick $n \in N_r$ with r maximal, note that this implies $\mathfrak{m}n \subseteq N_{r+1} \oplus N_{r+2} \oplus \ldots = 0$. The first isomorphism theorem on $A \longrightarrow An$, we see that $An \simeq A/\mathfrak{m} = k$. Consider the long exact sequence

$$0 \longrightarrow An \simeq k \longrightarrow M \longrightarrow M/An \longrightarrow 0$$

The long exact sequence in Tor

$$\ldots \operatorname{Tor}_3^A(k, M/An) \longrightarrow \operatorname{Tor}_2^A(k,k) \longrightarrow \operatorname{Tor}_2^A(k,M) \longrightarrow \ldots$$

Hilbert's syzygy theorem implies $\operatorname{Tor}_3^A(k, M/An) = 0$, also $\operatorname{Tor}_2^A(k,M) = 0$ but $\operatorname{Tor}_2^A(k,k) = k$, which contradicts exactness.