School of Mathematics University of New South Wales

MATH5725: Galois Theory (2011,S2)
Problem Set 6: Solvability by Radicals. Finite & Cyclotomic Fields [1]

1. Find the Galois groups of the following polynomials over $\mathbb{Q}$: i) $x^2 - 3$ ii) $x^4 - 2$ iii) $x^3 - 2$ iv) $x^5 - 4x + 2$ v) $x^4 - 8x^2 + 9$.

2. Using the definition or otherwise, show that $x^{12} + 6x^9 + 9x^6 - 3x^3 + 6$ is solvable by radicals.

3. Is $x^5 - 4x + 2 \in \mathbb{Q}[x]$ solvable by radicals?

4. Recall the following theorem (proof in Boris's notes on web): A finite field extension $K/\mathbb{Q}$ embeds in a radical extension if and only if its Galois group is solvable. Use this to show any quartic polynomial over $\mathbb{Q}$ is solvable. Show also that any Galois extension $K/\mathbb{Q}$ of degree 27 embeds in a radical extension.

5. Find all the primitive elements for the following field extensions: i) $\mathbb{Q}(\sqrt{2}, \sqrt{5})/\mathbb{Q}$ ii) $\mathbb{Q}(e^{2\pi i/3}, \sqrt[3]{5})/\mathbb{Q}$.

6. Let K be a finite field. Show that the multiplicative group of units K^* is cyclic.

7. Write down the lattice (i.e. set partially ordered by inclusion) of subfields of $\mathbb{F}_{3^{24}}$.

8. What is the Galois group of $\mathbb{F}_{1024}/\mathbb{F}_4$?

9. What is the smallest subfield of $\overline{\mathbb{F}_5}$ containing both $\mathbb{F}_{25}$ and $\mathbb{F}_{125}$?

10. Show that $\mathbb{Q}(i\sqrt{3})$ is the cyclotomic field of m-th roots of unity for some m.

11. Let $\zeta_m = e^{2\pi i/m}$. Show that $\mathbb{Q}(\zeta_{12})/\mathbb{Q}(\zeta_3)$ is Galois and compute its Galois group. Compute all the intermediate fields of $\mathbb{Q}(\zeta_{12})/\mathbb{Q}$ and hence $\mathbb{Q}(\zeta_3) \cap \mathbb{Q}(\zeta_4)$.

12. Can you construct using a ruler and compass a regular n-gon where: i) $n = 15$, ii) $n = 60$, iii) $n = 25$?

[1] by Daniel Chan