School of Mathematics and Statistics University of New South Wales

MATH5725: Galois Theory (2011,S2)
Problem Set 5: Radical extensions & Solvability [1]

1. Let K/F be a Galois extension with Galois group G. For any $\alpha \in K$, show that $f(x) := \Pi_{\sigma \in G}(x - \sigma(\alpha)) \in F[x]$ so the minimal polynomial of α over F divides $f(x)$. Use this to find the minimal polynomial of $2\sqrt{2} + \sqrt{3}$ over $\mathbb{Q}$ and over $\mathbb{Q}(\sqrt{2})$.

2. What are the Galois closures of the following field extensions? i) $\mathbb{Q}(\sqrt{2})/\mathbb{Q}$ ii) $\mathbb{Q}(\sqrt[3]{5})/\mathbb{Q}$ iii) $\mathbb{Q}(\sqrt{2+\sqrt{2}})/\mathbb{Q}$ iv) $\mathbb{Q}(\sqrt{3+\sqrt{2}})/\mathbb{Q}$.

3. Let p be the characteristic of a field F and n be a positive integer such that $p \nmid n$. Show that the group of n-th roots of unity $\mu_n < \bar{F}$ is cyclic of order n. Hint: use the structure of finite abelian groups to write μ_n as a product of cyclic groups.

4. Which of the following field extensions are radical? i) $\mathbb{Q}(\sqrt[3]{2})/\mathbb{Q}$ ii) $\mathbb{Q}(\sqrt{2}, \sqrt{7})/\mathbb{Q}$ iii) $\mathbb{Q}(\sqrt{3} - \sqrt{7})/\mathbb{Q}$ iv) $\mathbb{Q}(\sqrt[3]{2+\sqrt{2}})/\mathbb{Q}$ v) $\mathbb{F}_4/\mathbb{F}_2$ vi) $\mathbb{C}/\mathbb{R}$.

5. For the radical extensions in the previous question, write down a radical tower and the corresponding normal chain of subgroups with factors cyclic of prime order.

6. Show that any dihedral group is solvable. Compute the derived series of a dihedral group.

7. Show that the alternating group A_4 is solvable.

8. Is $S_5 \times S_3$ solvable?

9. Find the Sylow subgroups of A_4.

10. Find the Sylow subgroups of $\mathbb{Z}/8\mathbb{Z} \times \mathbb{Z}/18\mathbb{Z}$.

11. Let G be a group of order 88, with a normal subgroup of order 11. Show that G is solvable.

[1] by Daniel Chan

12. Show that S_4 has a subgroup

$$D = \{1, (12), (34), (13)(24), (12)(34), (14)(23), (1324), (1423)\}$$

which is isomorphic to the dihedral group of order 8.

13. (Harder) Let $c, d \in \mathbb{Q}$ and $K = \mathbb{Q}(\sqrt{c+\sqrt{d}}, \sqrt{c-\sqrt{d}})$. Suppose that $f(x) = x^4 - 2cx^2 + c^2 - d$ is irreducible and order the roots

$$\sqrt{c+\sqrt{d}}, -\sqrt{c+\sqrt{d}}, \sqrt{c-\sqrt{d}}, -\sqrt{c-\sqrt{d}}.$$

Show that Gal$K/\mathbb{Q}$ is a subgroup of the group D in the previous question. Show i) it is the cyclic group $\langle(1324)\rangle$ if $d(c^2-d)$ is a square in $\mathbb{Q}$, ii) it is $\{1, (13)(24), (12)(34), (14)(23)\}$ if $c^2 - d$ is a square in $\mathbb{Q}$ and finally iii) it is D otherwise. Hint: First determine all the possible Galois groups. Then pick $\sigma \in \operatorname{Gal} K/\mathbb{Q}$ which sends $\sqrt{c+\sqrt{d}} \mapsto \sqrt{c-\sqrt{d}}$ and examine the possibilities for $\sigma(c^2-d)$.