School of Mathematics University of New South Wales

MATH5725: Galois Theory (2011,S2)
Problem Set 4: Galois extensions & correspondence [1]

1. Which of the following field extensions is (finite) Galois? i) $\mathbb{Q}(\sqrt[3]{7})/\mathbb{Q}$ ii) $\mathbb{Q}(\sqrt{6})/\mathbb{Q}$ iii) $\mathbb{Q}(\sqrt[3]{7}, e^{2\pi i/3})/\mathbb{Q}$ iv) $\mathbb{Q}(\sqrt[3]{7}, e^{2\pi i/3})/\mathbb{Q}(e^{2\pi i/3})$ v) for prime p, $\mathbb{F}_p(\sqrt[p]{t})/\mathbb{F}(t)$ vi) $\mathbb{F}_9/\mathbb{F}_3$ vii) $\mathbb{C}/\mathbb{R}$ viii) $\mathbb{Q}(\sqrt{2}, \sqrt{3})/\mathbb{Q}$ ix) $\mathbb{Q}(\sqrt{2}, \sqrt{3})/\mathbb{Q}(\sqrt{2})$ x) $\mathbb{Q}(\sqrt[4]{2}, i)/\mathbb{Q}(i)$ xi) $\mathbb{Q}(\sqrt{2} + \sqrt{3})/\mathbb{Q}$.

2. What are the Galois groups of the field extensions in Q1?

3. Use the Galois correspondence to write down the "lattice" of all the intermediate fields of i) $\mathbb{Q}(\sqrt{2}, \sqrt{3})/\mathbb{Q}$ and ii) $\mathbb{Q}(\sqrt[4]{2}, i)/\mathbb{Q}(i)$.

4. Use the previous question to help you. Consider the field extension $K/F = \mathbb{Q}(\sqrt[4]{2}, i)/\mathbb{Q}(i)$. What are the following intermediate fields? i) $\mathbb{Q}(i, 3\sqrt{2} - 2 + 5i)$ ii) $\mathbb{Q}(i, \sqrt[4]{2} + i\sqrt{2} + 3\sqrt[4]{8})$.

5. Let $\omega = e^{2\pi i/3}$ and $K/F = \mathbb{Q}(\sqrt[3]{2}, \omega, \sqrt{5})/\mathbb{Q}(\omega)$. Compute the Galois group Gal K/F and the Galois correspondence. Hint: this example is similar to biquadratic extensions.

6. Show that $K := \mathbb{Q}(\sqrt[4]{2}, i)$ is the splitting field for $f(x) = x^4 - 2$ over $\mathbb{Q}$. Compute the Galois group of $K/\mathbb{Q}(i)$. Compute the Galois group of $K/\mathbb{Q}$. Write out the Galois correspondence and use it to determine the intermediate fields L with $L/\mathbb{Q}$ Galois. Hint: it may help to plot the roots of $f(x)$ on the Argand diagram and consider elements of the Galois group as permutations of the corners of the resulting quadrilateral.

7. Complete the example of lecture 9. Let σ, τ be automorphisms of the field $K = \mathbb{C}(x, y)$ defined by $(\sigma f)(x, y) = f(-x, y), (\tau f)(x, y) = f(x, -y)$. Show these are indeed automorphisms and compute all the intermediate fields of K/K^G where $G = \langle \sigma, \tau \rangle$.

[1] by Daniel Chan