School of Mathematics & Statistics University of New South Wales

# MATH5725: Galois Theory (2011,S2)
# Problem Set 3: Constructing Field Automorphisms. Normal & Separable Extensions [1]

1. Let $\omega$ be a primitive 5-th root of unity. Show that $F = \mathbb{Q}(\sqrt[5]{3}, \omega)$ is a splitting field for $f(x) = x^5 - 3$ over $\mathbb{Q}$. i) Show there exists a field automorphism $\sigma$ of $\mathbb{Q}(\sqrt[5]{3}, \omega)/\mathbb{Q}$ which sends $\sqrt[5]{3} \mapsto \sqrt[5]{3}\omega$. ii) Show there exists a field automorphism $\sigma$ of $\mathbb{Q}(\sqrt[5]{3}, \omega)/\mathbb{Q}$ which sends $\sqrt[5]{3}\omega \mapsto \sqrt[5]{3}\omega^{-1}$ but fixes $\sqrt[5]{3}$ (hint: look at the field extension $\mathbb{Q}(\sqrt[5]{3}, \omega)/\mathbb{Q}(\sqrt[5]{3})$).

2. Show that there exists a field homomorphism $\sigma : \mathbb{Q}(\sqrt[3]{2}) \longrightarrow \mathbb{C}$ over $\mathbb{Q}$ which maps $\sqrt[3]{2} \mapsto \sqrt[3]{2}e^{2\pi i/3}$. If $d$ is the separable degree of $\mathbb{Q}(\sqrt[3]{2})/\mathbb{Q}$ then what is $d$ and what are the $d$ field homomorphisms $\mathbb{Q}(\sqrt[3]{2}) \longrightarrow \overline{\mathbb{Q}}$?

3. Show that $\mathbb{Q}(\sqrt{3}, \sqrt{10})$ is a splitting field for $p(x) = (x^2 - 3)(x^2 - 10)$ over $\mathbb{Q}$. Show that $\mathrm{Gal}(\mathbb{Q}(\sqrt{3}, \sqrt{10})/\mathbb{Q}$ is isomorphic to a subgroup $G$ of the permutation group Perm $S$ where $S = \{\sqrt{3}, -\sqrt{3}, \sqrt{10}, -\sqrt{10}\}$. Use the uniqueness of splitting fields to find $G$. Hint: Show that $G$ has at most 4 elements and construct those field automorphisms individually.

4. Find $\mathrm{Gal}(\mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{7})/\mathbb{Q})$.

5. Which of the following are normal field extensions i) $\mathbb{Q}(\sqrt{6})/\mathbb{Q}$, ii) $\mathbb{Q}(\sqrt[3]{5})/\mathbb{Q}$, iii) $\mathbb{Q}(\sqrt[3]{5}, e^{2\pi i/3})/\mathbb{Q}$, iv) $\mathbb{Q}(\sqrt[3]{5}, e^{2\pi i/3})/\mathbb{Q}(\sqrt[3]{5})$,
v) $\mathbb{Q}(\sqrt[3]{5}, e^{2\pi i/3})/\mathbb{Q}(e^{2\pi i/3})$, vi) $\mathbb{F}_p(\sqrt[p]{t})/\mathbb{F}_p(t)$, vii) a degree 2 field extension $K/F$ if the characteristic of $F$ is not 2, viii) $\mathbb{Q}(\sqrt[6]{2})/\mathbb{Q}(\sqrt{2})$ ix) $\mathbb{Q}(e^{\pi i/3})/\mathbb{Q}$?

6. Which of the following extensions are separable? i) $\mathbb{C}/\mathbb{R}$, ii) $\mathbb{Q}(\sqrt[8]{11})/\mathbb{Q}$, iii) $\mathbb{F}_3(\sqrt{2})/\mathbb{F}_3$, iv) the splitting field for $x^2+x+1$ over $\mathbb{F}_3$, v) the splitting field for $x^9 + 2t$ over $\mathbb{F}_3(t)$.

7. Let $F \subseteq L \subseteq K$ be a tower of field extensions and $\alpha \in K$ be separable over $F$. Show that $\alpha$ is also separable over $L$.

8. Let $\alpha_1, \ldots, \alpha_n$ be separable elements over a field $F$. Show that the extension $F(\alpha_1, \ldots, \alpha_n)/F$ is also separable i.e. an extension which is finitely generated by separable elements is itself separable.

[1] by Daniel Chan