School of Mathematics University of New South Wales

MATH5725: Galois Theory (2011,S2)
Problem Set 2: Splitting fields [1]

Below, F will always denote a field.

1. Write down a splitting field for $x^3 - 1$ over $\mathbb{Q}$ in the form i) $\mathbb{Q}(\alpha)$ for some element $\alpha \in \mathbb{C}$ and ii) $\mathbb{Q}[x]/\langle p(x) \rangle$ for some polynomial $\mathbb{Q}[x]$.

2. Let $\mathbb{F}_3 = \mathbb{Z}/3\mathbb{Z}$ be the field with 3 elements. Write down a splitting field K for $x^3 - 1$ over $\mathbb{F}_3$. How many elements does K have?

3. Show that any splitting field of a polynomial $f(x) \in F[x]$ is a finite extension of F.

4. Write down a splitting field for $x^3 - 5$ over i) $\mathbb{Q}$ ii) $\mathbb{Q}(\sqrt[3]{5})$ iii) $\mathbb{R}$.

5. If $f(x) \in F[x]$ is a linear polynomial, what is the splitting field for $f(x)$ over F?

6. What is the splitting field for $x^3 - 6x$ over $\mathbb{R}$?

7. Do the exercise in lecture 2. Let $F \subseteq L \subseteq K$ be a tower of field extensions and $f(x) \in F[x]$. i) Show that if K is a splitting field for $f(x)$ over F, then it is also a splitting field for $f(x)$ over L. ii) Show the converse holds if L is generated over F by some of the roots of $f(x)$.

8. Repeat the previous question but for a family of polynomials $\{f_i(x)\} \subseteq F[x] - F$.

9. Show that the splitting field of a finite set $f_1(x), \ldots, f_n(x)$ of polynomials is the same as a splitting field for the single polynomial $f(x) = f_1(x)f_2(x)\ldots f_n(x)$.

10. Show that the algebraic closure of a field F is a splitting field for $F[x] - F$ over F.

11. What is a splitting field for $\{x^2 - 2, x^2 - 3, x^2 - 5\}$ over i) $\mathbb{Q}$ ii) $\mathbb{Q}(\sqrt{2})$ iii) $\mathbb{Q}(\sqrt[3]{7})$?

[1] by Daniel Chan

12. Show that if K is a splitting field for $f(x) \in F[x]$ over F, then it contains all the roots of $f(x)$ in any extension field L of K.

13. Let $F = \mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5}, \sqrt{7}, \ldots)$. i) For any positive rational number q, show that $\sqrt{q} \in F$. ii) Show that any subfield L of F which is finite over $\mathbb{Q}$ has degree a power of 2. iii) Hence or otherwise show that $\sqrt[3]{5} \notin F$.