MATH5725: Galois Theory (2011,S2)
Assignment 2. Due tuesday week 12 [1]

1. Let $\mathbb{F}_2$ be the field with two elements. Compute the following: i) $[\mathbb{F}_2(\sqrt[3]{t}) : \mathbb{F}_2(t)]$, ii) $[\mathbb{F}_2(\sqrt[3]{t}) : \mathbb{F}_2(t)]_s$ and iii) $[\mathbb{F}_2(\sqrt[6]{t}) : \mathbb{F}_2(t)]_s$.

2. In this question, you will need to play around with the symmetric group S_3. You may find it handy to use the fact that S_3 is generated by $\sigma = (123), \tau = (12)$ with defining relations $\sigma^3 = 1 = \tau^2, \tau\sigma = \sigma^{-1}\tau$ so in particular, every element can be written uniquely in the form $\sigma^i\tau^j$ for $i = 0, 1, 2$ and $j = 0, 1$.

   Let $G = \mathbb{Z}/3\mathbb{Z} \times S_3$.

   (a) Compute the commutators $[\sigma^i, \tau]$, $[\sigma^i, \sigma]$, $[\sigma^i\tau, \tau]$ and $[\sigma^i\tau, \sigma]$. Your answer is best written in the form $\sigma^m\tau^n$ for some m, n.

   (b) Determine the commutator subgroup $[G, G]$. Hint: You should be able to do this question without computing all commutators, just guess the answer and prove using properties given in lecture 13 that your answer is correct.

   (c) Compute the derived series for G and hence determine if G is solvable or not.

   (d) Using your answer in part i) or otherwise, compute $Z(G)$.

3. Let $K = \mathbb{Q}(\sqrt{2}, \sqrt{7}, \sqrt{11})$. In this question, you are free to assume that $\sqrt{7} \notin \mathbb{Q}(\sqrt{2})$ and that $\sqrt{11} \notin \mathbb{Q}(\sqrt{2}, \sqrt{7})$.

   (a) What is $[K : \mathbb{Q}]$?

   (b) Explain why $K/\mathbb{Q}$ is a Galois extension.

   (c) Compute the Galois group G of $K/\mathbb{Q}$ and show it is isomorphic to a product of cyclic groups. State what those cyclic groups are.

   (d) Compute all the subgroups of G of order 2.

   (e) HENCE find all intermediate fields L of $K/\mathbb{Q}$ such that $[K : L] = 2$. Be sure to invoke the Galois correspondence to ensure that you have all of them.

   (f) Use Galois theory to find the minimal polynomial of $\alpha = 2\sqrt{2} + \sqrt{7} + \sqrt{11}$ over $\mathbb{Q}(\sqrt{14})$.

[1] by Daniel Chan