School of Mathematics and Statistics University of New South Wales

MATH5725: Galois Theory (2011,S2) Assignment 1. Due thursday week 5 [1]

Below, F will always denote a field.

1. What is $[\mathbb{Q}(\sqrt{2}, \sqrt[5]{2}) : \mathbb{Q}]$? Write down a $\mathbb{Q}$-basis for $\mathbb{Q}(\sqrt{2}, \sqrt[5]{2})$.

2. Let K be the splitting field of $x^4 - 2x^2 - 1$ over $\mathbb{Q}$.
   (a) Write down K in the form $\mathbb{Q}(\alpha, \beta)$ for some roots α, β.
   (b) Compute $[K : \mathbb{Q}]$. (Hint: it may be easier to compute $[K : \mathbb{Q}(\alpha)]$ and $[\mathbb{Q}(\alpha) : \mathbb{Q}]$, noting $K \not\subseteq \mathbb{R}$).

3. Let $f(x), g(x) \in F[x] - F$. If L is a splitting field for $f(x)$ over F and K is a splitting field for $g(x)$ over L, show that K is also a splitting field for $f(x)g(x)$ over F.

4. Consider the field $K = \mathbb{Q}(e^{\pi i/4})$.
   (a) Is $K/\mathbb{Q}$ a normal field extension? Justify your answer fully.
   (b) Show that any element of the Galois group $G = \mathrm{Gal}K/\mathbb{Q}$ permutes the elements of the set $S = \{e^{\pi i/4}, e^{-\pi i/4}, e^{3\pi i/4}, e^{-3\pi i/4}\}$.
   (c) Show that the Galois group G has 4 elements, and write them down as permutations of S.
   (d) Any group of order 4 is isomorphic to either the cyclic group $\mathbb{Z}/4\mathbb{Z}$ or the Klein four group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Which of these groups is G isomorphic to? Justify your answer.

5. Let $\bar{F}$ be the algebraic closure of F and suppose that $[\bar{F} : F] = 2$. Show that every irreducible polynomial in $F[x] - F$ is either linear or quadratic. (Hint: Pick a root $\alpha \in \bar{F}$ of $f(x) \in F[x] - F$ and analyse the possibilities for $F(\alpha)$).

[1] by Daniel Chan