

THE UNIVERSITY OF NEW SOUTH WALES
SCHOOL OF MATHEMATICS AND STATISTICS

November 2009

MATH5725
GALOIS THEORY

- (1) TIME ALLOWED – 2 HOURS
- (2) TOTAL NUMBER OF QUESTIONS – 4
- (3) ATTEMPT ALL QUESTIONS
- (4) THE QUESTIONS ARE **NOT** OF EQUAL VALUE
- (5) THIS PAPER MAY BE RETAINED BY THE CANDIDATE

All answers must be written in ink. Except where they are expressly required pencils may only be used for drawing, sketching or graphical work.

1. [48 marks] Justify your answers with a brief explanation (but be careful to mention the key points).
 - a) [4 marks] Is \mathbb{C}/\mathbb{R} a Galois field extension?
 - b) [4 marks] What is the Galois group of $\mathbb{Q}(\sqrt{2}, \sqrt{3})/\mathbb{Q}$?
 - c) [4 marks] Find all the Sylow 3-subgroups of S_4 .
 - d) [4 marks] Is the Galois group of $x^6 - 3x^3 - 3$ over \mathbb{Q} isomorphic to S_6 ?
 - e) [4 marks] Is the group $S_3 \times S_3$ solvable?
 - f) [4 marks] What is the Galois closure of $\mathbb{Q}(\sqrt[5]{2})/\mathbb{Q}$?
 - g) [4 marks] Is $\mathbb{Q}(\sqrt{3} - \sqrt{7})/\mathbb{Q}$ a radical extension?
 - h) [4 marks] Let K/F be a Galois extension of degree 27. Show that there is an intermediate field L such that L/F is a Galois extension of degree 3.
 - i) [4 marks] True or false, the Galois group of an irreducible cubic in $\mathbb{Q}[x]$ is either cyclic of order 3 or the symmetric group S_3 .
 - j) [4 marks] Is $\mathbb{F}_{27}/\mathbb{F}_3$ separable?
 - k) [4 marks] What is $\text{Gal}(\mathbb{F}_{16}/\mathbb{F}_2)$?
 - l) [4 marks] Can you construct a regular 15-gon with a ruler and compass?
2. [10 marks] Let $K = \mathbb{Q}(\sqrt[4]{2}, i)$, $F = \mathbb{Q}(i)$.
 - a) [3 marks] What is the Galois group of K/F ?
 - b) [4 marks] Write out the Galois correspondence for the field extension K/F .
 - c) [3 marks] Is $\sqrt[4]{2} + i\sqrt{2}$ a primitive element for the field extension K/F ? Justify your answer.
3. [10 marks] Let $\zeta_9 = e^{2\pi i/9}$.
 - a) [2 marks] What is the minimal polynomial of ζ_9 over \mathbb{Q} ?
 - b) [4 marks] What is the Galois group of $\mathbb{Q}(\zeta_9)/\mathbb{Q}$? Is it cyclic?
 - c) [4 marks] Write down the Galois correspondence for the field extension $\mathbb{Q}(\zeta_9)/\mathbb{Q}$.
4. [12 marks] Let $f(x) = x^5 - 5x^4 + 5 \in \mathbb{Q}[x]$ and K be its splitting field over \mathbb{Q} .
 - a) [4 marks] Is $f(x)$ solvable by radicals over \mathbb{Q} ? Justify your answer fully.

- b) [4 marks] How many intermediate fields L of K/\mathbb{Q} are there with $[K : L] = 5$? Justify your answer fully.
- c) [4 marks] How many intermediate fields L of K/\mathbb{Q} have $[L : \mathbb{Q}] = 2$? Justify your answer fully.