THE UNIVERSITY OF NEW SOUTH WALES

SCHOOL OF MATHEMATICS AND STATISTICS

November 2007

MATH5725
GALOIS THEORY

(1) TIME ALLOWED – 3 HOURS

(2) TOTAL NUMBER OF QUESTIONS – 6

(3) ATTEMPT ALL QUESTIONS

(4) THE QUESTIONS ARE **NOT** OF EQUAL VALUE

(5) THIS PAPER MAY BE RETAINED BY THE CANDIDATE

All answers must be written in ink. Except where they are expressly required pencils may only be used for drawing, sketching or graphical work.

1. (36 marks total) Justify your answers with a brief explanation (but be careful to mention the key points).

a) Let K/F be a Galois extension of degree n. What is the order of the Galois group Gal K/F?

b) What is Gal $\mathbb{F}_{64}/\mathbb{F}_2$?

c) Is the Galois group of $x^6 - 3x^3 - 6$ over $\mathbb{Q}$ isomorphic to S_6?

d) Is the dihedral group of order 10 solvable?

e) What is the Galois closure of $\mathbb{Q}(\sqrt[5]{2})/\mathbb{Q}$?

f) Let $L/K, K/F$ be Galois field extensions.

i) Is L/F separable?

ii) Is L/F Galois?

g) Is $x^5 - 4x + 2 \in \mathbb{Q}[x]$ solvable by radicals?

h) Give an example of a Galois extension K of $\mathbb{Q}(i)$ whose Galois group is isomorphic to $\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

i) For $K = \mathbb{Q}(\sqrt{2}, \sqrt{3})$, compute $N_{K/\mathbb{Q}}(2 + \sqrt{2})$.

j) Let K/F be a finite abelian extension and L be an intermediate field. Is L/F a Galois extension?

k) Let $L/K, K/F$ be purely inseparable field extensions. Is L/F purely inseparable?

2. (8 marks total) Let $K = \mathbb{Q}(e^{2\pi i/3}, \sqrt[3]{5})$ and consider the field extension $K/\mathbb{Q}$.

a) (3 marks) What is the Galois group G of $K/\mathbb{Q}$? Justify your answer.

b) (5 marks) Write down all the intermediate fields of $K/\mathbb{Q}$ and the corresponding subgroups of G. You need not explain your computations.

3. (8 marks) Let K/F be a finite Galois extension with Galois group $G \simeq G_1 \times G_2$ for some groups G_1, G_2. Show that there exist intermediate fields L_1, L_2 satisfying the following three conditions.

a) For $i = 1, 2$, L_i/F is Galois with Galois group isomorphic to G_i.

b) K is the smallest field containing both L_1 and L_2.

c) $L_1 \cap L_2 = F$.

Make sure you fully justify your answer.

4. (5 marks total) Consider the field automorphism $\sigma : \mathbb{C}(t) \longrightarrow \mathbb{C}(t)$ defined by

$$(\sigma f)(t) := f(t^{-1}).$$

Let G be the cyclic group generated by σ.

a) (1 mark) What is the order of G?

b) (2 marks) What is $[\mathbb{C}(t) : \mathbb{C}(t + t^{-1})]$?

c) (2 marks) Prove that $\mathbb{C}(t)^G = \mathbb{C}(t + t^{-1})$.

5. (6 marks total) This question concerns the existence of the so called "maximal abelian subextension".

a) (3 marks) Let G be a (not necessarily Hausdorff) topological group and N be a normal subgroup. Prove that the closure of N is also a normal subgroup.

b) (3 marks) Let K/F be a Galois extension. Show that there exists an intermediate field K_{ab} satisfying the following two conditions.

i) K_{ab}/F is an abelian extension.

ii) If L is any intermediate field of K/F such that L/F is also an abelian extension, then $L \subseteq K_{ab}$.

6. (7 marks total) Recall that we have the following pro-finite groups

$$\hat{\mathbb{Z}} = \varprojlim_n \mathbb{Z}/n\mathbb{Z} \quad , \quad \hat{\mathbb{Z}}_p = \varprojlim_j \mathbb{Z}/p^j\mathbb{Z}$$

where p is any prime.

a) (3 marks) Show that

$$\hat{\mathbb{Z}} \simeq \prod_{p,\text{ prime}} \hat{\mathbb{Z}}_p.$$

b) (2 marks) Let $\phi : \hat{\mathbb{Z}} \longrightarrow \hat{\mathbb{Z}}_p$ denote the natural projection. Show that $H := \ker \phi$ is a closed subgroup of $\hat{\mathbb{Z}}$.

c) (2 marks) Suppose q is a prime and let us identify the absolute Galois group $\mathrm{Gal}\ \overline{\mathbb{F}}_q/\mathbb{F}_q$ with $\hat{\mathbb{Z}}$ as in lectures so that we may consider H as a subgroup of $\mathrm{Gal}\ \overline{\mathbb{F}}_q/\mathbb{F}_q$. Determine with reason, the fixed field $\overline{\mathbb{F}}_q^H$.