School of Mathematics University of New South Wales

MATH5725: Galois Theory (2016,S2)
Problem Set: Graphs [1]

This problem sheet covers lectures on Graphs. Recall that an isomorphism of graphs is a morphism which is bijective on both the set of vertices and edges.

1. Let X be a set of abstract generators and consider a set of relations $w_i = w'_i, i \in I$ where I is some index set and $w_i, w'_i \in W(X)$ are words in $X^{\pm 1}$. We define the *group generated by X with defining relations $w_i = w'_i$* to be $G = F(X)/N$ where N is the intersection of all normal subgroups of $F(X)$ containing the subgroup generated by all the $w_i^{-1}w'_i$. We denote it by $\langle x \in X | w_i = w'_i \rangle$. Prove that $\langle x, y | x^2 = 1, y^n = 1, yx = xy^{-1} \rangle$ is isomorphic to the dihedral group by establishing a suitable universal property or otherwise.

2. Find the automorphism group of the graph $B\mathbb{Z}$.

3. Let Γ, Δ be graphs. The *Cartesian product* of Γ and Δ is the graph $\Gamma \square \Delta$ whose set of vertices is $\Gamma_0 \times \Delta_0$ and edges are those of the form $(v, w) \xrightarrow{\delta_{v\alpha}} (v, w')$ where $v \in \Gamma_0, w \xrightarrow{\alpha} w' \in \Delta_1$ or $(v, w) \xrightarrow{\delta_{w\beta}} (v', w)$ where $w \in \Delta_0, v \xrightarrow{\beta} v' \in \Gamma_1$. (I hope the $\overline{(\cdot)}$ operation is clear!). Draw $B\mathbb{Z} \square B\mathbb{Z}$.

4. Let $p : \tilde{\Gamma} \longrightarrow \Gamma$ be a cover of (conected) graphs and $v \in \tilde{\Gamma}_0$. Show that any deck transformation σ of p is completely determined by $\sigma(v)$. Hint: Use path lifting property of covers.

5. Let $X = \{x, y\}$ and $S^1 \vee S^1$ be the graph with one vertex and two directed edge pairs $x^{\pm 1}, y^{\pm 1}$. Show that $F(X) \setminus BF(X)$ is isomorphic to $S^1 \vee S^1$.

6. Construct a graph morphism $q : B\mathbb{Z} \square BZ \longrightarrow S^1 \vee S^1$ which makes q a cover. What's the corresponding Galois group?

7. Show that there is an intermediate cover of the form $BF(X) \longrightarrow B\mathbb{Z} \square B\mathbb{Z} \longrightarrow S^1 \vee S^1$. Can you work out the corresponding subgroup of the Galois group $F(X)$?

[1] by Daniel Chan