School of Mathematics University of New South Wales

MATH5725: Galois Theory (2016,S2)
Problem Set: Galois cohomology [1]

This problem sheet covers lectures on infinite Galois theory.

1. Recall that $G = \mathrm{Gal}(\mathbb{C}/\mathbb{R})$ acts on the group $\mathbb{C}^*$ and hence on the subgroup μ_n by restriction where n is any positive integer. Compute the cohomology group $H^1(G, \mu_n)$.

2. Let $\zeta = e^{2\pi i/5}$ and $K = \mathbb{Q}(\zeta)$. Compute $N_{K/\mathbb{Q}}(\cos(2\pi/5))$.

3. Without resorting to the quartic formula, show that every quartic over $\mathbb{Q}$ is solvable by radicals.

4. Let σ be the generator for $G = \mathrm{Gal}(\mathbb{C}/\mathbb{R})$. Show that the 1-cocycle α_* of G with values in $PGL_2(\mathbb{C})$ determined by $\alpha_\sigma = \left(\begin{smallmatrix} 0 & i \\ -i & 0 \end{smallmatrix}\right)$ is not cohomologous to the trivial cocycle.

5. Suppose a finite group G acts on groups A and B and there is a group homomorphism $\phi : A \longrightarrow B$ which is compatible with the G-action, that is $\phi(g.a) = g.\phi(a)$ for all $g \in G, a \in A$. Show that ϕ induces a set map $H^1(G, A) \longrightarrow H^1(G, B)$ which preserves the trivial cocycle.

6. A *(non-degenerate) quadratic F-space* is a finite dimensional vector space V over F equipped with a symmetric bilinear form $(-,-) : V \times V \longrightarrow F$ which is non-degenerate in the sense that if $(v, -)$ is zero then $v = 0$. Two quadratic F-spaces V, W are isomorphic if there is an F-linear isomorphism $\phi : V \longrightarrow W$ with $(f(v), f(v')) = (v, v')$ for all $v, v' \in V$. If F is algebraically closed, show that any two quadratic F-spaces of the same dimension are isomorphic.

7. The *standard* quadratic space structure on the K-space K^n is given by the bilinear form $(v, w) = v^T w$. Find a quadratic space structure on $\mathbb{R}^2$ which is not isomorphic to the standard one.

8. Let K/F be a finite Galois extension. An F-rational form of a quadratic K-space V is an F-rational form W of the underlying vector space such that the K-bilinear form on V restricts to an F-bilinear form on W, i.e. $(w, w') \in F$ for all $w, w' \in W$. Let V be a quadratic F-space. Show that there is a finite Galois extension such that V is isomorphic to an F-rational form of K^n with the standard quadratic space structure.

[1] by Daniel Chan

9. Let K/F be a finite Galois extension. Let $O_n(K)$ be the *orthogonal group* $O_n(K) = \{M \in GL_n(K) | M^T M = I_n\}$ and note that $G = \mathrm{Gal}(K/F)$ acts on this group entry-wise. Show that the F-rational forms of K^n are given by 1-cocycles of G with values in $O_n(K)$ and that two such are isomorphic if and only if they are cohomologous i.e. isomorphism classes of F-rational forms are classified by elements of the cohomology set $H^1(G, O_n(K))$.