School of Mathematics University of New South Wales

MATH5725: Galois Theory (2016,S2)
Problem Set 5: Solvability by Radicals. Finite fields [1]

This problem sheet covers lectures 16-18 and completes the basic material in Galois theory.

1. Find the Galois groups of the following polynomials over $\mathbb{Q}$: i) x^2-3 ii) x^4-2 iii) x^3-2 iv) x^5-4x+2 v) x^4-8x^2+9 vi) x^3+2x-2.
2. Using the definition or otherwise, show that $x^{12}+6x^9+9x^6-3x^3+6$ is solvable by radicals.
3. Is $x^5-4x+2 \in \mathbb{Q}[x]$ solvable by radicals?
4. Is $2x^5-5x^4+5 \in \mathbb{Q}[x]$ solvable by radicals?
5. Verify the formula for the discriminant of a cubic given in lectures.
6. Let S_n act on $K = \mathbb{Q}(\alpha_1, \ldots, \alpha_n)$ by permuting the α_i and $\delta = \prod_{i<j}(\alpha_i - \alpha_j)$. Determine the Galois group of $K/K^G(\delta)$.
7. HD Using the Vandermonde determinant formula or otherwise, show that the discriminant of a polynomial $f(x)$ with roots $\alpha_1, \ldots, \alpha_n$ (counted with multiplicity) is

$$\begin{vmatrix} p_0 & p_1 & \cdots & p_{n-1} \\ p_1 & p_2 & & p_n \\ \vdots & & \ddots & \vdots \\ p_{n-1} & p_n & \cdots & p_{2n-2} \end{vmatrix}$$

where $p_i = \sum_j \alpha_j^i$.

8. Write down the poset of subfields of $\mathbb{F}_{3^{24}}$.
9. What is the Galois group of $\mathbb{F}_{1024}/\mathbb{F}_4$?
10. What is the smallest subfield of $\overline{\mathbb{F}_5}$ containing both $\mathbb{F}_{25}$ and $\mathbb{F}_{125}$?
11. HD Let F be a field of characteristic $p > 0$ and K be an extension. Suppose that $\beta \in F$ is a root of $f(x) = x^p - x - \alpha$ where $\alpha \in F$. Show that $\beta + 1$ is also a root of $f(x)$. Determine the Galois group of $F(\beta)/F$.

[1] by Daniel Chan

12. HHD Let F be a field of characteristic $p > 0$. If $\alpha \in F$ is not a p-th power in F, show that $x^p - \alpha$ is irreducible. Hint: Factorise $x^p - \alpha$ in the splitting field first.

13. HHD Let F be a field of characteristic $p > 0$. We say that F is *perfect* if every finite field extension is separable. Prove that F is perfect if and only if the Frobenius homomorphism on F is an isomorphism.