School of Mathematics & Statistics UNSW, Australia

MATH5725: Galois Theory (2016, S2)
Problem Set 4: Radical extensions & solvable groups [1]

This problem set covers material in lectures 11-15. Again most of the questions are aimed for Pass to Distinction level students. Those aimed at HD students have been marked.

1. What are the Galois closures of the following field extensions? i) $\mathbb{Q}(\sqrt{2})/\mathbb{Q}$ ii) $\mathbb{Q}(\sqrt[3]{5})/\mathbb{Q}$ iii) HD $\mathbb{Q}(\sqrt{2+\sqrt{2}})/\mathbb{Q}$ iv) HD $\mathbb{Q}(\sqrt{3+\sqrt{2}})/\mathbb{Q}$.

2. Find all primitive elements of the following field extensions: i) $\mathbb{Q}(\sqrt{5},\sqrt{7})/\mathbb{Q}$, ii) HD $\mathbb{C}(x)/\mathbb{C}(x^4)$.

3. Which of the following field extensions are radical? i) $\mathbb{Q}(\sqrt[3]{2})/\mathbb{Q}$ ii) $\mathbb{Q}(\sqrt{2},\sqrt{7})/\mathbb{Q}$ iii) HD $\mathbb{Q}(\sqrt{3}-\sqrt{7})/\mathbb{Q}$ iv) $\mathbb{Q}(\sqrt[3]{2+\sqrt{2}})/\mathbb{Q}$ v) $\mathbb{F}_4/\mathbb{F}_2$ vi) $\mathbb{C}/\mathbb{R}$.

4. HD For the radical Galois extensions in the previous question, write down a radical tower and the corresponding normal chain of subgroups with factors cyclic of prime order.

5. HD Recall from Problem Set 3 that $K := \mathbb{Q}(\sqrt[4]{2}, i)$ is a Galois extension of $\mathbb{Q}$. Find a the radical tower for $K/\mathbb{Q}$ and the corresponding normal chain of subgroups with cyclic factors.

6. HD Show that any dihedral group is solvable. Compute the derived series of a dihedral group.

7. HD Show that the alternating group A_4 is solvable.

8. Is $S_5 \times S_3$ solvable?

9. Show $S_3 \times S_2$ is solvable by producing a normal chain of subgroups with cyclic factors.

10. HD Find the Sylow subgroups of A_4.

11. Find the Sylow subgroups of $\mathbb{Z}/8\mathbb{Z} \times \mathbb{Z}/18\mathbb{Z}$.

12. Find a normal chain of subgroups of the quaternion group which has cyclic factors of order 2.

[1] by Daniel Chan

13. Find the centre of the dihedral group.

14. HHD Prove that any non-abelian group G of order 8 is isomorphic to the dihedral group or the quaternion group as follows: i) Show that G must contain an element g of order 4, ii) pick $h \in G - \langle g \rangle$ and show that $h^2 = 1$ or g^2, iii) Show that in the first case we get the dihedral group whilst in the second, we get the quaternion (Hint: study the conjugation action of $\langle h \rangle$ on $\langle g \rangle$).

15. HHD Prove that there are no groups with $[G : Z(G)] = 2$.

16. HD Prove the following partial converse to Lagrange's theorem. Let G be a p-group. For any divisor d of $|G|$, there is a subgroup of G with order d.

17. HD Let G be a group of order 88, with a normal subgroup of order 11. Show that G is solvable.

18. HHD Show that any group of order 21 is solvable. HHHD Determine all groups of order 21 up to isomorphism.