School of Mathematics & Statistics UNSW, Australia

MATH5725: Galois Theory (2016, S2)
Problem Set 3: Galois correspondence [1]

This problem set covers material in lectures 6-10. Again most of the questions are aimed for Pass to Distinction level students. Those aimed at HD students have been marked.

1. Check the following field extensions are Galois and write down the Galois correspondence for them i.e. determine a) the Galois group G, b) the poset of subgroups of G, c) the corresponding poset of intermediate fields and d) the normal subgroups of G

   i) $\mathbb{Q}(\sqrt{3}, \sqrt{5})/\mathbb{Q}$, ii) $\mathbb{Q}(\sqrt{3}, \sqrt{5})/\mathbb{Q}(\sqrt{5})$ iii) $\mathbb{Q}(\sqrt[4]{2}, i)/\mathbb{Q}(i)$

   iv) $\mathbb{Q}(\sqrt{2} + \sqrt{3})/\mathbb{Q}$, v) $\mathbb{Q}(\sqrt[3]{7}, e^{2\pi i/3})/\mathbb{Q}$

2. HD Let $\omega = e^{2\pi i/3}$ and $K = \mathbb{Q}(\sqrt[3]{2}, \omega)$. Show that $\sqrt{5} \notin K$. Hint: Assume by way of contradiction that $\sqrt{5} \in K$ and use the Galois correspondence for $K/\mathbb{Q}$ to determine which intermediate field $\mathbb{Q}(\sqrt{5})$ must be.

3. HD Let $\omega = e^{2\pi i/3}$ and $K/F = \mathbb{Q}(\sqrt[3]{2}, \omega, \sqrt{5})/\mathbb{Q}(\omega)$. Compute the Galois group Gal K/F and the Galois correspondence. Hint: this example is similar to biquadratic extensions.

4. HD Show that $K := \mathbb{Q}(\sqrt[4]{2}, i)$ is the splitting field for $f(x) = x^4 - 2$ over $\mathbb{Q}$. Compute the Galois group G of $K/\mathbb{Q}(i)$. Compute the Galois group of $K/\mathbb{Q}$. Write out the Galois correspondence and use it to determine the intermediate fields L with $L/\mathbb{Q}$ Galois. Determine the orbits of G on both the poset of subgroups of G and the poset of intermediate fields. Hint: it may help to plot the roots of $f(x)$ on the Argand diagram and consider elements of the Galois group as permutations of the corners of the resulting quadrilateral.

5. Let σ, τ be automorphisms of the field $K = \mathbb{C}(x, y)$ defined by $(\sigma f)(x, y) = f(-x, y), (\tau f)(x, y) = f(x, -y)$. Show these are indeed automorphisms satisfying $\sigma^2 = \tau^2 = 1, \sigma\tau = \tau\sigma$. Compute all the intermediate fields of K/K^G where $G = \langle \sigma, \tau \rangle$.

[1] by Daniel Chan

6. HD Show that S_4 has a subgroup

$$D = \{1, (12), (34), (13)(24), (12)(34), (14)(23), (1324), (1423)\}$$

which is isomorphic to the dihedral group of order 8.

7. HHD Let $c, d \in \mathbb{Q}$ and $K = \mathbb{Q}(\sqrt{c+\sqrt{d}}, \sqrt{c-\sqrt{d}})$. Suppose that $f(x) = x^4 - 2cx^2 + c^2 - d$ is irreducible and order the roots

$$\sqrt{c+\sqrt{d}}, -\sqrt{c+\sqrt{d}}, \sqrt{c-\sqrt{d}}, -\sqrt{c-\sqrt{d}}.$$

Show that $G := \mathrm{Gal} K/\mathbb{Q}$ is a subgroup of the group D in the previous question. Show i) it is the cyclic group $\langle (1324) \rangle$ if $d(c^2-d)$ is a square in $\mathbb{Q}$, ii) it is $\{1, (13)(24), (12)(34), (14)(23)\}$ if $c^2 - d$ is a square in $\mathbb{Q}$ and finally iii) it is D otherwise. Hint: First determine all the possible Galois groups. If $|G| = 4$, show that the intermediate field $\mathbb{Q}(\sqrt{d})$ corresponds to the subgroup generated by $(12)(34)$. Show $\sqrt{c^2-d} \in \mathbb{Q}(\sqrt{d})$. Then pick $\sigma \in \mathrm{Gal}\, K/\mathbb{Q}$ which sends $\sqrt{c+\sqrt{d}} \mapsto \sqrt{c-\sqrt{d}}$ and show $\sigma(\sqrt{c^2-d}) = \pm\sqrt{c^2-d}$ depending on whether you are in case i) or ii).

8. Show that $\mathbb{Q}(i\sqrt{3})$ is the cyclotomic field of m-th roots of unity for some m.

9. Let $\zeta = e^{\pi i/5}$. Show that $G = \mathrm{Gal}\, \mathbb{Q}(\zeta)/\mathbb{Q}$ is generated by $\sigma : \zeta \mapsto \zeta^3$. Determine the Galois correspondence for $\mathbb{Q}(\zeta)/\mathbb{Q}$. What is the minimal polynomial of ζ over $\mathbb{Q}$. HD What is the minimal polynomial of $\cos(\pi/5)$ over $\mathbb{Q}$? Determine $\mathbb{R} \cap \mathbb{Q}(\zeta)$. Is the regular 10-gon constructible?

10. Let $\zeta = e^{2\pi i/7}$. HD Determine the Galois correspondence for $\mathbb{Q}(\zeta)/\mathbb{Q}$. What is the minimal polynomial of ζ over $\mathbb{Q}$. Determine $\mathbb{R} \cap \mathbb{Q}(\zeta)$. Is the regular 7-gon constructible?

11. Let $\zeta_m = e^{2\pi i/m}$. Show that $\mathbb{Q}(\zeta_{12})/\mathbb{Q}(\zeta_3)$ is Galois and compute its Galois group. Compute all the intermediate fields of $\mathbb{Q}(\zeta_{12})/\mathbb{Q}$ and hence $\mathbb{Q}(\zeta_3) \cap \mathbb{Q}(\zeta_4)$.

12. Can you construct using a ruler and compass a regular n-gon where: i) $n = 15$, ii) $n = 60$, iii) $n = 25$?