School of Mathematics & Statistics UNSW, Australia

MATH5725: Galois Theory (2016, S2)
Problem Set 2: Galois group basics [1]

This problem set covers material in lectures 1-5. most of the questions are aimed for Pass to Distinction level students. Those aimed at HD students have been marked.

1. Write down a splitting field for $x^3 - 1$ over $\mathbb{Q}$ in the form i) $\mathbb{Q}(\alpha)$ for some element $\alpha \in \mathbb{C}$ and ii) $\mathbb{Q}[x]/\langle p(x)\rangle$ for some polynomial $\mathbb{Q}[x]$.

2. Show that $\mathbb{Q}(\sqrt{3}, \sqrt{10})$ is a splitting field for $p(x) = (x^2 - 3)(x^2 - 10)$ over $\mathbb{Q}$. Show that $\mathrm{Gal}(\mathbb{Q}(\sqrt{3}, \sqrt{10})/\mathbb{Q}$ is isomorphic to the subgroup $G = \langle(1\ 2), (3\ 4)\rangle$ of S_4.

3. Using the previous question, determine the minimal polynomials of the following numbers over $\mathbb{Q}$: i) $3 - \sqrt{3}$, ii) $2 - 3\sqrt{3} + \sqrt{30}$.

4. Find $\mathrm{Gal}(\mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{7})/\mathbb{Q})$.

5. Find $\mathrm{Gal}(\mathbb{Q}(\sqrt[3]{7}, e^{2\pi i/3})/\mathbb{Q})$.

6. Let $K = \mathbb{C}(x, y)$ the field of rational functions in x and y. Consider the following maps $\sigma, \tau : K \longrightarrow K$

$$(\sigma f)(x, y) = f(-x, y), \quad (\tau f)(x, y) = f(x, -y).$$

   i) What is $\sigma(x^2)$? ii) Show that σ, τ are automorphisms of K. iii) Show that $\sigma^2 = 1$ and $\sigma\tau = \tau\sigma$. iv) Determine the group G generated by σ and τ. v) Compute the fixed field K^G. vi) What is the minimal polynomial of $\frac{x}{y}$ over K^G?

7. Let ω be a primitive 5-th root of unity and $F = \mathbb{Q}(\sqrt[5]{3}, \omega)$. i) Show there exists a field automorphism σ of $\mathbb{Q}(\sqrt[5]{3}, \omega)/\mathbb{Q}$ which sends $\sqrt[5]{3} \mapsto \sqrt[5]{3}\omega$. ii) HD Show there exists a field automorphism τ of $\mathbb{Q}(\sqrt[5]{3}, \omega)/\mathbb{Q}$ which sends $\sqrt[5]{3}\omega \mapsto \sqrt[5]{3}\omega^2$ but fixes $\sqrt[5]{3}$. iii) What is $\tau(1 + \sqrt[5]{3}\omega^3)$?

8. HD Let K/F be the splitting field of an irreducible polynomial $f(x)$ over F. Show that its Galois group G acts transitively on the roots of $f(x)$ i.e. given any two roots α, β of $f(x)$, there exists $\sigma \in G$ with $\sigma(\alpha) = \beta$.

[1] by Daniel Chan

9. HD Let $\alpha \in \mathbb{C}$ be a root of $f(x) = x^3 + x^2 - 2x - 1$. Show that $\alpha^2 - 2$ is also a root of $f(x)$ and determine $\mathrm{Gal}(\mathbb{Q}(\alpha)/\mathbb{Q})$.

10. Show that any splitting field of a polynomial $f(x) \in F[x]$ is a finite extension of F.

11. Let $\mathbb{F}_3 = \mathbb{Z}/3\mathbb{Z}$ be the field with 3 elements. Write down a splitting field K for $x^3 - 1$ over $\mathbb{F}_3$. How many elements does K have?

12. Let $F \subseteq L \subseteq K$ be a tower of field extensions and $f(x) \in F[x]$. i) Show that if K is a splitting field for $f(x)$ over F, then it is also a splitting field for $f(x)$ over L. ii) Show the converse holds if L is generated over F by some of the roots of $f(x)$.

13. Which of the following are normal field extensions and which are Galois? i) $\mathbb{Q}(\sqrt{6})/\mathbb{Q}$, ii) $\mathbb{Q}(\sqrt[3]{5})/\mathbb{Q}$, iii) $\mathbb{Q}(\sqrt[3]{5}, e^{2\pi i/3})/\mathbb{Q}$, iv) $\mathbb{Q}(\sqrt[3]{5}, e^{2\pi i/3})/\mathbb{Q}(\sqrt[3]{5})$, v) $\mathbb{Q}(\sqrt[3]{5}, e^{2\pi i/3})/\mathbb{Q}(e^{2\pi i/3})$, vi) $\mathbb{F}_p(\sqrt[p]{t})/\mathbb{F}_p(t)$, vii) a degree 2 field extension K/F if the characteristic of F is not 2, viii) $\mathbb{Q}(\sqrt[6]{2})/\mathbb{Q}(\sqrt{2})$?

14. Prove Proposition-Definition 1 in lecture 5.

15. Which of the following extensions are separable? i) $\mathbb{C}/\mathbb{R}$, ii) $\mathbb{Q}(\sqrt[8]{11})/\mathbb{Q}$, iii) $\mathbb{F}_3(\sqrt{2})/\mathbb{F}_3$, iv) the splitting field for $x^3 - x$ over $\mathbb{F}_3$, v) the splitting field for $x^9 + 2t$ over $\mathbb{F}_3(t)$.