School of Mathematics & Statistics UNSW, Australia

MATH5725: Galois Theory (2016, S2)
Problem Set 1: revision [1]

The purpose of this problem set is to make sure you are familiar with all the basic notions you learnt in MATH3711, MATH5706 or its equivalent elsewhere. You should be familiar with the following concepts from:

- **Field theory**: characteristic of a field, algebraic extensions, finite extensions, simple field extensions, degree of a field extension, tower of field extensions, minimal polynomial.
- **Group theory**: normal subgroups, quotient groups, isomorphism theorems, permutation groups, abelian groups.
- **Ring theory**: irreducible elements, factor theorem, polynomial ring, Gauss's lemma.

I will assume that you will be on top of these notions by the end of week 2. Please come and see me if you need help with this as this course will be extremely difficult otherwise.

1. Which of the following rings are fields: $\mathbb{R}, \mathbb{Q}, \mathbb{C}, \mathbb{R}[x], \mathbb{R}(x), \mathbb{Z}, \mathbb{Z}/5\mathbb{Z}, \mathbb{Z}/6\mathbb{Z}, \mathbb{Z}[i]/\langle 2+i\rangle, \mathbb{Q}[\sqrt{2}], \mathbb{Q}[\pi], \mathbb{Q}(\pi), M_3(\mathbb{R})$?
2. What are the characteristics of $\mathbb{C}, \mathbb{R}(x), \mathbb{F}_4, \mathbb{F}_{27}, \mathbb{Z}/5\mathbb{Z}, \mathbb{Z}[i]/\langle 2+i\rangle, \mathbb{Q}[\sqrt{2}]$?
3. Write down a $\mathbb{Q}$-basis for the field $\mathbb{Q}(\sqrt[3]{2})$ and determine the degree $[\mathbb{Q}(\sqrt[3]{2}) : \mathbb{Q}]$ of the field extension $\mathbb{Q}(\sqrt[3]{2})/\mathbb{Q}$.
4. Is $\alpha := \sqrt{2+\sqrt{5}}$ algebraic over $\mathbb{Q}$? If so, determine its minimal polynomial and hence the degree of $\mathbb{Q}(\alpha)/\mathbb{Q}$.
5. Let E/F be a finite field extension of prime degree. Show that E is a simple extension of F.
6. Is the field extension $\mathbb{Q}(\sqrt{2}, \sqrt[3]{3})/\mathbb{Q}$ finite, and if so, what is its degree and a $\mathbb{Q}$-basis? Is it algebraic?
7. Prove that a field extension K/F, generated by a finite number of algebraic elements is finite.

[1] by Daniel Chan

8. Suppose $K/L, L/F$ are field extensions of degrees 2 and 3 respectively. Is K/F finite and if so, what is its degree? Is K/F algebraic?

9. What is the degree of $\mathbb{Q}(i, \sqrt[4]{3})/\mathbb{Q}$? Write down a $\mathbb{Q}$-basis for $\mathbb{Q}(i, \sqrt[4]{3})$.

10. Show that $V = \{1, (12)(34), (13)(24), (14)(23)\}$ is a normal subgroup of S_4. What well-known group is it isomorphic to?

11. Find all subgroups of $\mathbb{Z}/12\mathbb{Z}$.

12. Show that $x^3 - 3x + 1$ is irreducible over $\mathbb{Q}$.