Lecture 7: Proof of Fundamental Thm

Aim Lecture: Finish proof fund. thm.

Artin's Lemma

Lemma (Artin): Let H = finite group of autom. of a field K. Then
$$[K : K^H] \leq |H|$$

Proof: Let $H = \{1 = \sigma_1, \sigma_2, \ldots, \sigma_n\}$ & suppose by way of contradiction that $\alpha_1, \ldots, \alpha_{n+1} \in K$ are lin. indep. $/K^H$.

Consider $n \times (n+1)$-matrix $/K$

$$A = \begin{pmatrix} \sigma_1(\alpha_1) & \ldots & \sigma_1(\alpha_{n+1}) \\ \vdots & & \\ \sigma_n(\alpha_1) & \ldots & \sigma_n(\alpha_{n+1}) \end{pmatrix} \qquad \begin{pmatrix} x_1 \\ \vdots \\ x_{n+1} \end{pmatrix} = \underset{\sim}{x}$$

where $\sigma_1(\alpha_1) = \alpha_1$, $\sigma_1(\alpha_{n+1}) = \alpha_{n+1}$.

no. col > no. rows $\Rightarrow \exists$ non-zero $\underset{\sim}{x} \in K^{n+1}$ with $A\underset{\sim}{x} = \underset{\sim}{0}$. Pick such a sol$^{\underline{n}}$ with max. no. zero entries. Contradiction will follow on constructing another $\underset{\sim}{x}$ with more 0's.

Permuting col. & scaling, can assume $x_1 = 1$. Now 1st row of $A = (\alpha_1 \ldots \alpha_{n+1})$

so lin. indep. of $\{\alpha_i\}/K^H \Rightarrow$ can sim. assume $x_2 \notin K^H$ i.e. $\exists\, \sigma \in H$ with $\sigma x_2 \neq x_2$

Apply σ to lin. eq$^{\underline{n}}$s $A\underset{\sim}{x} = \underset{\sim}{0}$

$$\underbrace{\begin{pmatrix} \sigma\sigma_1(\alpha_1) & \cdots & \sigma\sigma_1(\alpha_{n+1}) \\ \vdots & & \\ \sigma\sigma_n(\alpha_1) & \cdots & \sigma\sigma_n(\alpha_{n+1}) \end{pmatrix}}_{=\,\sigma A} \underbrace{\begin{pmatrix} \sigma x_1 \\ \vdots \\ \sigma x_{n+1} \end{pmatrix}}_{=\,\sigma \underset{\sim}{x}} = \underset{\sim}{0}$$

But σA is A with rows permuted so $A\sigma\underset{\sim}{x} = \underset{\sim}{0}$ too.

$$\therefore \quad A(\underbrace{\underset{\sim}{x} - \sigma\underset{\sim}{x}}_{=\,\underset{\sim}{x}'}) = \underset{\sim}{0}$$

We check $\underset{\sim}{x}'$ non-zero with more 0 entries than $\underset{\sim}{x}$.

$x'_1 = x_1 - \sigma(x_1) = 1 - \sigma(1) = 0$

$x_2' = x_2 - \sigma(x_2) \neq 0$

$x_i = 0 \Rightarrow x_i' = x_i - \sigma(x_i) = 0$

$\square$

Cor: (a) K/K^H is a finite Galois ext$^{\underline{n}}$

(b) $\mathrm{Gal}(K/K^H) = H$

(c) $[K:K^H] = H$

Proof: (a) Artin's lemma $\Rightarrow$ can find finite basis $\alpha_1, \dots, \alpha_m$ for K/K^H.

Lect. 2 thm $\Rightarrow$ min. poly $p_i(x)$ of α_i $/K^H$ is separable & factorises into linears $/K$. $\therefore$ K = splitting field of sep. poly. $p(x) = p_1(x) \dots p_m(x) / K^H$.

(b) & (c) Note H fixes K^H so $H \leq \mathrm{Gal}(K/K^H)$. $\therefore$

$$|H| \leq |\mathrm{Gal}(K/K^H)| \overset{\text{Lect. 5 cor \& (a)}}{=} [K:K^H] \overset{\text{lemma}}{\leq} |H|$$

$\therefore$ Have equality $\Rightarrow H = \mathrm{Gal}(K/K^H)$ $\square$

$\boxed{\text{Proof Fundamental Thm}}$

Thm: K/F = Galois ext$^{\underline{n}}$ Galois group G. Galois correspondence below bijective

$$L \longmapsto L' = \mathrm{Gal}(K/L)$$

$$\{\text{Inter. fields of } K/F\} \rightleftarrows \{\text{subgroups of } G\}$$

$$H' := K^H \longleftarrow\!\!\!\mapsto H$$

Proof: Saw $L'' = L$ so suffice show $H'' = H$. But
$$H'' = (K^H)' = \mathrm{Gal}(K/K^H) \overset{\text{Gr}}{=} H.$$

Example with Function Fields

$$K = \mathbb{C}(x)$$

ex. Check $\sigma : K \to K : f(x) \mapsto f(ix)$ is a field autom.

$\sigma(x^4) = (ix)^4 = x^4 \Rightarrow x^4 \in K^\sigma$

Note $(\sigma^2 f)(x) = (\sigma f)(ix) = f(i^2 x) = f(-x)$

$\therefore\ (\sigma^4 f)(x) = (\sigma^2 f)(-x) = f(-(-x)) = f(x)$

ie. $G = \langle \sigma \rangle$ cyclic order 4.

Cor. $\Rightarrow$ K/K^G Galois deg. 4.

ex. Galois corresp. here is

$$\begin{array}{ccc}
\mathbb{Z}/4 \simeq \langle \sigma \rangle & & K^{\langle \sigma \rangle} = \mathbb{C}(x^4) \\
\vee & & \cap \\
\mathbb{Z}/2 \simeq \langle \sigma^2 \rangle & & K^{\langle \sigma^2 \rangle} = \mathbb{C}(x^2) \\
\vee & & \cap \\
1 & & \mathbb{C}(x)
\end{array}$$