Lecture 2: Fixed fields

Aim Lecture: See example of how symmetry throws light on field theory. Key concept intro is the fixed field.

Field Automorphisms

K = field

Def$^{\underline{n}}$ 1: A field automorphism of K is a ring isomorphism of form $\sigma: K \to K$.

Some examples include

Eg 1. $\sigma: \mathbb{C} \to \mathbb{C}: z \mapsto \bar{z}$ conjugation

Why?

E.g. 2, $\sigma_1: \mathbb{Q}(\sqrt{2}, \sqrt{3}) \longrightarrow \mathbb{Q}(\sqrt{2}, \sqrt{3})$

$\mathbb{Q}(\sqrt{2}, \sqrt{3}) = [\mathbb{Q}(\sqrt{2})](\sqrt{3})$

$\alpha_i \in \mathbb{Q}$ $\quad \underbrace{\alpha_1 + \alpha_2\sqrt{2}}_{\beta_1} + \underbrace{\alpha_3\sqrt{3} + \alpha_4\sqrt{6}}_{=\beta_2\sqrt{3}} \longmapsto \alpha_1 + \alpha_2\sqrt{2} - \alpha_3\sqrt{3} - \alpha_4\sqrt{6}$

$\beta_1 + \beta_2\sqrt{3} \longmapsto \beta_1 - \beta_2\sqrt{3}$

$\beta_i \in \mathbb{Q}(\sqrt{2})$

$\sigma_2: \mathbb{Q}(\sqrt{2}, \sqrt{3}) \longrightarrow \mathbb{Q}(\sqrt{2}, \sqrt{3})$

$\alpha_1 + \alpha_2\sqrt{2} + \alpha_3\sqrt{3} + \alpha_4\sqrt{6} \longmapsto \alpha_1 - \alpha_2\sqrt{2} + \alpha_3\sqrt{3} - \alpha_4\sqrt{6}$

Ex: check!

Prop$^{\underline{n}}$ 1: For field K, the set $\mathrm{Aut}\, K$ of all field automorphisms of K is a subgroup of the permutation group on K. In particular, any subgroup of $\mathrm{Aut}\, K$ acts on K.

Proof: Just check closure axioms.
e.g. closed under mult$^{\underline{n}}$ $\because$ composites of ring isom. are ring isom.

Fixed fields

K = field.

Prop$^{\underline{n}}$-Def$^{\underline{n}}$: Let $S \subseteq \operatorname{Aut} K$. The fixed field of K wrt S is

$$K^S := \{\alpha \in K \mid \sigma(\alpha) = \alpha \quad \forall \sigma \in S\}.$$

It is a subfield of K.

Proof: Just check closure axioms e.g.
closure under add$^{\underline{n}}$: $\alpha, \beta \in K^S$
$\sigma(\alpha+\beta) = \sigma(\alpha) + \sigma(\beta) = \alpha + \beta \quad \forall \sigma \in S$
$\Rightarrow \alpha + \beta \in K^S$ too.

Prop$^{\underline{n}}$ 2: ⓐ If $S' \subseteq S \subseteq \operatorname{Aut} K \Rightarrow$
$K^S \subseteq K^{S'}$.
ⓑ If $\langle S \rangle$ = group gen. by S then
$K^S = K^{\langle S \rangle}$.

Proof: ex.
Hint ⓑ $\sigma_1, \sigma_2 \in S$, $\alpha \in K^S \Rightarrow$
$(\sigma_1^{\pm 1} \sigma_2^{\pm 1})(\alpha) = \alpha$

Eg. 1 again $\sigma : \mathbb{C} \to \mathbb{C}$ conj$^{\underline{n}}$
gen. group order 2.
$\mathbb{C}^\sigma = \{ z \in \mathbb{C} : \bar{z} = z \} = \mathbb{R}$.

Eg. 2 again $\sigma_1, \sigma_2 : \mathbb{Q}(\sqrt{2}, \sqrt{3}) \circlearrowleft$

$$\mathbb{Q}(\sqrt{2}, \sqrt{3})^{\sigma_1} = \mathbb{Q}(\sqrt{2})$$
$$\mathbb{Q}(\sqrt{2}, \sqrt{3})^{\langle \sigma_1, \sigma_2 \rangle} = \mathbb{Q}$$

Using Symmetry to compute minimal polynomials

Prop$^{\underline{n}}$ 3: Let $\varphi : R \to S$ be a ring hom. Then $\varphi[x] : R[x] \to S[x]$

$$p(x) = \sum r_i x^i \longmapsto \sum \varphi(r_i) x^i =: (\varphi p)(x)$$

is a ring hom.

Proof: ex. checking axioms.

Thm: Let G be a finite subgroup of $\operatorname{Aut} K$ & $\alpha \in K$. Let $\alpha_1 = \alpha, \alpha_2, \ldots, \alpha_n$ be the orbit of α under action of G on K. Then

$$p(x) = (x - \alpha_1)(x - \alpha_2) \ldots (x - \alpha_n)$$

is the min. poly. of α / K^G.

Proof: Note $p(\alpha) = 0$. To check $p(x) \in K^G[x]$ it suffices to note $\sigma \in G \Rightarrow$

$$(\sigma p)(x) = (x - \sigma(\alpha_1)) \ldots (x - \sigma(\alpha_n))$$
$$= p(x)$$

so all co-eff. are fixed by σ.

Hence the monic min. poly. $q(x)$ of α

divides $p(x)$.

$q(x) \in K^G[x] \Rightarrow \quad (\sigma q)(x) = q(x) \ \forall \sigma \in G$

Hence $q(x)$ contains all the linear factors of $p(x)$ so $p(x) = q(x)$ $\square$

Eg. 1 again min poly of $\alpha \in \mathbb{C}$ over $\mathbb{R} = \mathbb{C}^{\langle\sigma\rangle}$ is

$$\begin{cases} (x-\alpha)(x-\bar{\alpha}) & \text{if } \alpha \neq \bar{\alpha} \text{ i.e. } \alpha \notin \mathbb{R} \\ x-\alpha & \text{if } \alpha \in \mathbb{R}. \end{cases}$$

Eg. 2 again. ex. Show σ_1, σ_2 gen. Klein 4 group $\{1, \sigma_1, \sigma_2, \sigma_1\sigma_2\} \simeq \mathbb{Z}/2 \times \mathbb{Z}/2$.

The min poly. of $3 + 2\sqrt{2} + \sqrt{3}$ $/\mathbb{Q}$ is

$$(x-3-2\sqrt{2}-\sqrt{3})(x-3-2\sqrt{2}+\sqrt{3})(x-3+2\sqrt{2}-\sqrt{3})\cdot(x-3+2\sqrt{2}+\sqrt{3})$$