Lecture 14: Insolvable groups & p-groups

Aim Lecture: Collect some results on group to give examples of (in)solvable groups & applications

Some Insolvable Groups

Prop^{n}1: For $n \geq 5$, the alternating group A_5 is not solvable.

Proof: Lect. 13 Cor. 2 $\Rightarrow$ suffice show by induction on r that $A_n^{(r)}$ contains all 3-cycles $(a\ b\ c)$.

Note $(a\ b\ c)$ even so $r=0$ case OK.

$n \geq 5 \Rightarrow$ can pick $d, e \in \{1, \dots, n\}$ so a, b, c, d, e distinct & inductive hypothesis $\Rightarrow$

$$A_n^{(r+1)} = [A_n^{(r)}, A_n^{(r)}] \ni [(a\ d\ b), (a\ e\ c)]$$
$$= (a\,d\,b)^{-1}(a\,e\,c)^{-1}(a\,d\,b)(a\,e\,c)$$
$$= (a\ b\ c) \qquad \square$$

E.g. S_5 is not solvable $\because$ subgroup A_5 isn't.

Review Group Actions

Let G = finite group, X = set &

$\varphi: G \longrightarrow \text{Perm}\, X$ (= group of perm^{n}s of X)

be a group hom. Then G acts on X by

$g.x = [\varphi(g)](x)$ for $g \in G, x \in X$

E.g. 1. Conjugation action of G on G

$$\mathrm{conj}: G \longrightarrow \mathrm{Perm}\, G$$
$$g \longmapsto (c_g : h \longmapsto ghg^{-1})$$

Facts: ⓐ X is a disjoint union of its orbits $G.x = \{g.x \mid g \in G\}$

ⓑ $|G.x| = [G : \mathrm{stab}_G x]$ where $\mathrm{stab}_G x = \{g \in G \mid g.x = x\} \leq G$.

Propⁿ-Defⁿ: The centre $Z(G)$ of G consists of those elts $z \in G$ satisfying the following equivalent condⁿs

ⓐ $gz = zg \ \forall g \in G$ ⓑ $gzg^{-1} = z \ \forall g \in G$

ⓒ $zgz^{-1} = g \ \forall g \in G$.

$Z(G) \trianglelefteq G$.

Proof: Clear ⓐ $\Leftrightarrow$ ⓑ $\Leftrightarrow$ ⓒ.

ⓒ means $Z(G) = $ kernel of conjugation map $\mathrm{conj}: G \to \mathrm{Perm}\, G$ above so $Z(G) \trianglelefteq G$.

$\square$

Rem 1: Recall the fixed points of G acting on X is $X^G = \{x \in X \mid g.x = x \ \forall g\}$. ⓑ means $Z(G)$ is the fixed point set of conjugⁿ action.

$\boxed{\text{p-groups}}$ Let p = prime.

Defⁿ 1: ⓐ A group P is a p-group if $|P| = p^r$ for some $r \in \mathbb{N}$.

ⓑ Let $P \leq G$ where G is a group of order $p^r q$, $\gcd(p,q)=1$. We say P is a Sylow p-subgroup if $|P|=p^r$ i.e. P is a p-group & $[G:P]$ is coprime to p.

E.g. 2. $G=S_3$ has order 6.
Sylow 3-subgroup is $\langle(1\,2\,3)\rangle$
Sylow 2-subgroups are $\langle(1\,2)\rangle, \langle(1\,3)\rangle, \langle(2\,3)\rangle$.

Lemma: Let P be a p-group acting on a finite set X. Then $|X| \equiv |X^P| \mod p$.
Proof: Suffice show $p \mid |X - X^P|$. Now X^P = union of all 1-pt orbits. All other orbits $P.x$ have order $[P : \mathrm{Stab}_P x]$ which is a multiple of p $\because$ $\mathrm{Stab}_P x$ is a proper subgroup of p-group P. $\square$

Cor: If $P \neq 1$ is a p-group then $Z(P) \neq 1$.
Proof: Apply Lemma to conjug$^{\underline{n}}$ action & use Rem1.
Prop$^{\underline{n}}$ 2: (a) Any p-group P is solvable.
(b) In part., P has a normal chain of subgroups with factors $\mathbb{Z}/p\mathbb{Z}$.
Proof: Lagrange + (a) $\Rightarrow$ (b).
Prove (a) by induction on $|P|$.
Now Cor. $\Rightarrow$ $1 \neq Z(P) \trianglelefteq P$ so the inductive hypothesis $\Rightarrow$ $P/Z(P)$ is solvable. Also $Z(P)$ is solvable being abelian. $\therefore$ P is solvable $\square$

Groups of Order 8

Abelian groups of order 8 are
$\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$, $\mathbb{Z}/4\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$ & $\mathbb{Z}/8\mathbb{Z}$.

Let G = group gen. by a, b satisfying relations

$$(*) \qquad a^2=b^2, \quad a^4=1, \quad ba=a^3b$$

so G has at most 8 elts:

$$(\dagger) \quad G=\{1, a, a^2, a^3, b, ab, a^2b, a^3b\}.$$

Defn 2: We say such a group G is a quaternion group if $|G|=8$.

E.g. 3

$$G=\left\langle a=\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, b=\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right\rangle < GL_2(\mathbb{C})$$

is quaternion $\because$ $a^2=b^2=\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$

$$a^3b=\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}=\begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}=ba$$

& 8 elts in $(\dagger)$ are distinct.

Rem: The relns $(*)$ completely determine how to multiply 2 elts in $(\dagger)$ so any 2 quaternion groups are isomorphic.

In problem sets we'll show

Thm: Any nonabelian group of order 8 is either quaternion or dihedral.