Lecture 13: Solvable Groups

Aim Lecture: Study cond^{n} of solvability

Subgroups & Quotients

Prop^{n} 1: Let G = finite group, $H \leq G$, $N \trianglelefteq G$. Then

ⓐ G solvable $\Rightarrow$ H solvable

ⓑ G is solvable iff N & G/N are solvable

Proof: ⓑ($\Leftarrow$) Suppose

$$1 = N_0 \triangleleft N_1 \triangleleft \dots \triangleleft N_r = N$$

$$1_{G/N} = G_0/N \triangleleft G_1/N \triangleleft \dots \triangleleft G_s/N = G/N$$

are normal chains of subgroups with factors of prime order. Then

$$1 = N_0 \triangleleft N_1 \triangleleft \dots \triangleleft N_r = N = G_0 \triangleleft G_1 \triangleleft \dots \triangleleft G_s$$

is a normal chain of subgroups with factors N_{i+1}/N_i & $G_{i+1}/G_i \cong \dfrac{G_{i+1}/N}{G_i/N}$

so also have prime order factors.

$\therefore$ G is solvable

ⓑ($\Rightarrow$) & ⓐ Suppose G solvable with normal chain of subgroups

$$1 = G_0 \triangleleft G_1 \triangleleft G_2 \triangleleft \dots \triangleleft G_n = G$$

whose factors have prime order.

Check H solvable: by examining

chain of subgroups

$(*)\quad 1 = G_0 \cap H < G_1 \cap H < \dots < G_n \cap H = H.$

Chain normal $\because g \in G_{i+1} \cap H \Rightarrow$

$$g(G_i \cap H)g^{-1} = \underbrace{gG_ig^{-1}}_{\substack{= \\ G_i \,\because\, G_i \trianglelefteq G_{i+1}}} \cap \, g\underbrace{Hg^{-1}}_{\substack{= \\ H \,\because\, g \in H}} = G_i \cap H.$$

Recall isom. thm $JK/K \simeq J/J \cap K$

for $J \leq G_{i+1}$, $K \trianglelefteq G_{i+1}$.

$\therefore$ Factors of $(*)$ are

$$\frac{\overbrace{G_{i+1} \cap H}^{J}}{G_i \cap H} = \frac{G_{i+1} \cap H}{(G_{i+1} \cap H) \cap \underbrace{G_i}_{=K}} = \frac{(G_{i+1} \cap H)G_i}{G_i} \leq \frac{G_{i+1}}{G_i}$$

are subgroups of a cyclic group of prime order so are trivial or prime order.

$\therefore$ H is solvable.

Check G/N solvable:

Let $\pi : G \longrightarrow G/N : g \longmapsto gN$ be quotient map. We have chain of subgroups

$$(\dagger)\quad 1 = \underset{\substack{\shortparallel \\ G_0N/N}}{\pi(G_0)} \leq \underset{\substack{\shortparallel \\ G_1N/N}}{\pi(G_1)} \leq \dots \leq \pi(G_n) = G/N.$$

Chain normal $\because g \in G_{i+1} \Rightarrow$

$$\pi(g)\pi(G_i)\pi(g)^{-1} = \pi(gG_ig^{-1}) = \pi(G_i).$$

Suffice now check factors of (†) are trivial or cyclic prime order which follows from

Lemma 1: There's a surj. group hom

$$\bar{\pi}: G_{i+1}/G_i \longrightarrow \pi(G_{i+1})/\pi(G_i)$$
$$gG_i \longmapsto \pi(g)\pi(G_i)$$

Proof: Note $\bar{\pi}$ well-defined.
π hom $\Rightarrow \bar{\pi}$ is an hom. Surj. clear.

Cor 1: A finite group G is solvable iff it has a normal chain of subgroups

$$1 = G_0 \triangleleft G_1 \triangleleft \dots \triangleleft G_n = G$$

with abelian factors.

Proof: by induction on n. Inductive hypothesis $\Rightarrow G_{n-1}$ solvable & G/G_{n-1} solvable by Lect. 12 Prop^n 2. Prop^n 1 $\Rightarrow$ G solvable too. $\square$

$\boxed{\text{Derived Series}}$

G = group

Def^n 1: The commutator of $g, h \in G$ is $[g,h] := g^{-1}h^{-1}gh$. The commutator subgroup $[G,G]$ of G is the subgroup of G gen. by these commutators.

Lemma: ⓐ $C := [G,G] \trianglelefteq G$

(b) If $N \trianglelefteq G$ then G/N is abelian iff $N \geq C$.

(c) In particular, $G/[G,G]$ is abelian.

Proof: (a) ex & (b) $\Rightarrow$ (c).

(b). G/N abelian $\iff$ $\underset{\substack{\| \\ ghN}}{gNhN} = \underset{\substack{\| \\ hgN}}{hNgN}$ $\forall g,h \in G$

$\iff g^{-1}h^{-1}ghN = N$

$\iff [g,h] \in N$

$\iff [G,G] \leq N$.

Def2: The derived series of G is the infinite normal chain of subgroups

$$G = G^{(0)} \trianglerighteq G^{(1)} \trianglerighteq G^{(2)} \trianglerighteq \dots \qquad (**)$$

where $G^{(i+1)} = [G^{(i)}, G^{(i)}]$

Cor 2: A finite group G is solvable iff $G^{(n)} = 1$ for some n.

Proof: ($\Leftarrow$) follows from Cor. 1 & Lemma (i) since $(**)$ is a normal chain of subgroups with abelian factors.

($\Rightarrow$) Consider normal chain of subgroups

$$G = G_0 \triangleright G_1 \triangleright G_2 \triangleright \dots \triangleright G_n = 1$$

with abelian factors. We prove by induction on r that $G^{(r)} \leq G_r$ so $G^{(n)} = 1$ too.

G_r/G_{r+1} abelian $\Rightarrow$

$$G_{r+1} \geq [G_r, G_r] \geq [G^{(r)}, G^{(r)}] = G^{(r+1)}.$$

$\square$