Lecture 12: Galois groups of radical ext$\underline{\text{n}}$s

Aim Lecture: We study Galois groups of radical ext$\underline{\text{n}}$s to determine which polynomials are solvable by radicals.

To simplify discussion, we only study radical ext$\underline{\text{n}}$s & solvability in char 0 so assume this lecture F = field char. 0.

Let μ_n = group of n-th roots of unity in some splitting field L for $x^n - 1 / F$.
Lect. 9 Rem. $\Rightarrow$ μ_n cyclic order n.

$\boxed{\text{Simple Case}}$

Prop$\underline{\text{n}}$ 1: Let p = prime, F = field where $x^p - 1$ splits (ie. factorises into linears).
Let $\alpha \in F$ & $K = F(\sqrt[p]{\alpha})$. Then

(a) K is the splitting field for $x^p - \alpha / F$.

(b) If $\sqrt[p]{\alpha} \notin F$ then

$$\varphi : \mathrm{Gal}(K/F) \longrightarrow \mu_p \simeq \mathbb{Z}/p\mathbb{Z}$$
$$\sigma \longmapsto \frac{\sigma(\sqrt[p]{\alpha})}{\sqrt[p]{\alpha}}$$

is a well-defined group isom.

Proof: (a) $\mu_p \subset F \Longrightarrow$ the roots $\sqrt[p]{\alpha}\,\omega$, $\omega \in \mu_p$ all lie in $K = F(\sqrt[p]{\alpha})$ so K is the splitting field.

(b) $\sigma(\sqrt[p]{\alpha})$ is a root of $x^p - \alpha$ so $\frac{\sigma(\sqrt[p]{\alpha})}{\sqrt[p]{\alpha}} \in \mu_p$ & the map φ is well-defined.

We check φ is a hom.

$$\varphi(\sigma\tau)\sqrt[p]{\alpha} = (\sigma\tau)(\sqrt[p]{\alpha}) = \sigma\left(\sqrt[p]{\alpha}\,\varphi(\tau)\right)$$

$\mu_p \subset F$

$$= \sigma(\sqrt[p]{\alpha})\,\varphi(\tau) = \sqrt[p]{\alpha}\,\varphi(\sigma)\varphi(\tau).$$

$\therefore\ \varphi(\sigma\tau) = \varphi(\sigma)\varphi(\tau).$

Check φ isom.: $\sigma(\sqrt[p]{\alpha})$ determines σ so φ is injective. Lagrange $\Rightarrow$ suffice show $\operatorname{im}\varphi \neq 1$. But $\sqrt[p]{\alpha} \notin F$ $\Rightarrow$ $1 \neq [K:F] = |\mathrm{Gal}(K/F)|$ & we're done. $\square$

E.g. 1 $\mathrm{Gal}\left(\mathbb{Q}(\sqrt[3]{2}, \omega = e^{2\pi i/3}) / \mathbb{Q}(\omega)\right) \simeq \mu_3 \simeq \mathbb{Z}/3\mathbb{Z}$.
This also follows from Galois corresp. for $\mathbb{Q}(\sqrt[3]{2}, \omega)/\mathbb{Q}$ where Galois group corresp. to $\langle (1\ 2\ 3)\rangle$.

Solvability Galois corresp. & Prop$^{\underline{n}}$ suggest

Def$^{\underline{n}}$ 1: Let G = group. A normal chain of subgroups is a sequence of form

$$1 = G_0 \trianglelefteq G_1 \trianglelefteq G_2 \trianglelefteq \dots \trianglelefteq G_n = G.$$

The quotients G_{i+1}/G_i are called factors. We say G is solvable if there's such a chain with all factors cyclic of prime order (or trivial).

Eg.2. S_3 has normal chain of subgroups
$$1 \triangleleft A_3 = \langle (1\,2\,3) \rangle \triangleleft S_3$$
with factors $A_3 \simeq \mathbb{Z}/3\mathbb{Z}$, $S_3/A_3 \simeq \mathbb{Z}/2\mathbb{Z}$.
$\Longrightarrow S_3$ solvable.

Eg.3. Any cyclic group $G = \mathbb{Z}/n\mathbb{Z}$ is solvable $\because$ given prime factor$^{\underline{n}}$ $n = p_1 p_2 \cdots p_r$ have normal chain of subgroups
$$\mathbb{Z}/p_1 \cdots p_n\mathbb{Z} \triangleright p_1\mathbb{Z}/p_1 \cdots p_n\mathbb{Z} \triangleright p_1p_2\mathbb{Z}/p_1 \cdots p_n\mathbb{Z} \triangleright$$
$$\cdots \quad p_1p_2 \cdots p_{n-1}\mathbb{Z}/p_1 \cdots p_n\mathbb{Z}.$$
& Isom. thm $\Longrightarrow$ factors are
$$\frac{p_1 \cdots p_{i-1}\mathbb{Z}/n\mathbb{Z}}{p_1 \cdots p_i \mathbb{Z}/n\mathbb{Z}} \simeq \frac{p_1 \cdots p_{i-1}\mathbb{Z}}{p_1 \cdots p_i\mathbb{Z}} \simeq \mathbb{Z}/p_i\mathbb{Z}$$

Thm: Consider radical tower of field ext$^{\underline{n}}$s
$$F = F_0 \subseteq F_1 \subseteq F_2 \subseteq \cdots \subseteq F_n = K$$
say with $F_{i+1} = F_i(\sqrt[p_i]{\alpha_i})$ for some $\alpha_i \in F_i$ & p_i prime. Suppose all $x^{p_i} - 1$ split in F & K/F is Galois. Then $G = Gal(K/F)$ is solvable.

Proof: Galois corresp. gives chain of subgroups
$$(*)\ G = F_0' \geq F_1' \geq \cdots \geq F_n' = 1$$
K/F Galois $\Rightarrow$ K/F_i Galois.

Also Prop$^{\underline{n}}$ 1 $\Rightarrow$ F_{i+1}/F_i Galois with Galois group either isom. to $\mathbb{Z}/p_i\mathbb{Z}$ or 1.

Addendum to Fund. Thm in Lect. 10
$\Rightarrow$ $F_{i+1}' = \mathrm{Gal}(K/F_{i+1}) \trianglelefteq \mathrm{Gal}(K/F_i) = F_i'$
& furthermore $F_i'/F_{i+1}' \simeq \mathrm{Gal}(F_{i+1}/F_i)$.
$\therefore$ ⊛ is a normal chain of subgroups with all factors prime order or trivial.

$\square$

Eg. 2 again. Applying Galois corresp. to radical tower
$$\mu_2 \subseteq \mathbb{Q} \subseteq \mathbb{Q}(\sqrt{2}) \subseteq \mathbb{Q}(\sqrt{2}, \sqrt{3})$$
gives normal chain of subgroups
$$\underset{\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}}{\langle (12), (34)\rangle} \triangleright \underset{\mathbb{Z}/2\mathbb{Z}}{\langle (34)\rangle} \triangleright 1$$
with factors $\mathbb{Z}/2\mathbb{Z}$.

Prop$^{\underline{n}}$ 2: (a) G, H solvable $\Rightarrow$ $G\times H$ solvable
(b) In particular, all abelian groups are solvable.

Proof: Consider normal chains of subgroups with prime order factors
$$1 = G_0 \triangleleft G_1 \triangleleft \cdots \triangleleft G_m = G$$
$$1 = H_0 \triangleleft H_1 \triangleleft \cdots \triangleleft H_n = H.$$
$$1 \triangleleft G_1 \times 1 \triangleleft G_2 \times 1 \triangleleft \cdots \triangleleft G_m \times 1 \triangleleft G_m \times H_1 \triangleleft \cdots \triangleleft G \times H$$
is a normal chain of subgroups with same factors.

$\square$