Lecture 10: Galois action on Galois Correspondence

Aim Lecture: Extract more info from Galois corresp. by studying action of Galois group on it.

Galois Action Fix following not$^{\underline{n}}$ for this lecture

K/F = finite Galois ext$^{\underline{n}}$

G = Galois group.

Have Galois corresp

$$L \longmapsto L' = Gal(K/L)$$

$$\mathcal{F} := \left\{ \begin{array}{l} \text{intermediate} \\ \text{fields of } K/F \end{array} \right\} \rightleftarrows \left\{ \begin{array}{l} \text{subgroups} \\ \text{of } G \end{array} \right\} =: \mathcal{G}$$

$$H' = K^H \longleftarrow\!\!\shortmid\ H$$

G acts on $\mathcal{F}$ by: $\sigma \in G$, $L \in \mathcal{F} \Rightarrow \sigma(L) \in \mathcal{F}$ so σ permutes interm. fields.

Also G acts on $\mathcal{G}$ by conjug$^{\underline{n}}$ $\sigma : H \mapsto \sigma H \sigma^{-1}$.

Prop$^{\underline{n}}$ 1: $\sigma(L)' = \sigma L' \sigma^{-1}$

Proof: ($\supseteq$) $\sigma L' \sigma^{-1}$ fixes $\sigma(L)$ $\because$ given $\tau \in L' = Gal(K/L)$, $\alpha \in L$:

$$\sigma\tau\sigma^{-1}(\sigma(\alpha)) = \sigma\underbrace{\tau(\alpha)}_{\alpha} = \sigma(\alpha) \quad \because \tau \in L'$$

($\subseteq$) Apply ($\supseteq$) to $\sigma(L)$ & σ^{-1} to see $L' \supseteq \sigma^{-1}\sigma(L)'\sigma \Rightarrow \sigma L' \sigma^{-1} \supseteq \sigma(L)'$.

$\square$

Cor: $H \le G$ is normal iff $\sigma(K^H) = K^H$ for all $\sigma \in G$

Proof: Propⁿ $\Rightarrow$ fixed points of G action on $\mathcal{G}$ correspond to fixed points on $\mathcal{F}$.

Normality

Lemma: Let L be an intermediate field of K/F s.t. $\sigma(L) = L$ for all $\sigma \in G$. Then L/F is Galois.

Proof: Let $\alpha_1, .., \alpha_n$ be a set of gen. for L/F e.g. an F-basis. Let $p_1, \dots, p_n$ be their min. poly. $/F$.

Lect. 2 Thm $\Rightarrow$ each p_i factorises $p_i(x) = (x-\beta_1)\dots(x-\beta_r)$ where $\beta_1, .., \beta_r$ is the G-orbit α_i. Hence $p(x) = p_1(x)p_2(x)\dots p_n(x)$ is sep. Also $\sigma(L) = L$ $\Rightarrow$ all $\beta_j \in L$ so L is the splitting field of $p(x)/F$. $\square$

Addendum to Fundamental Thm

Let K/F be Galois, $G = Gal(K/F)$ & L = interm. field.

ⓐ L/F is Galois iff $L' = Gal(K/L) \trianglelefteq G$.

ⓑ In this case $Gal(L/F) \simeq G/L' = \dfrac{Gal(K/F)}{Gal(K/L)}$.

Proof: (a) Lemma + Cor. give ($\Leftarrow$) so we prove ($\Rightarrow$). L = splitting field of sep. $f(x) \in F[x]$ so $L = F(\alpha_1, .., \alpha_n)$ where $\alpha_1, .., \alpha_n$ are the roots of $f(x)$.

Any $\sigma \in G$ permutes the α_i so $\sigma(L) = L$.

Cor. + Galois corresp. $\Rightarrow L' \trianglelefteq G$.

(b) If $\sigma \in G$, then Cor $\Rightarrow$ the restriction $\sigma|_L$ of σ to L has image L so gives an isom. $\sigma|_L : L \xrightarrow{\sim} L$.

We thus obtain a group hom.

$$\varphi : Gal(K/F) \longrightarrow Gal(L/F)$$
$$\sigma \longmapsto \sigma|_L$$

Note $\ker\varphi = \{\sigma \in G \mid \underbrace{\sigma|_L = id}_{\text{means } \sigma \text{ fixes } L}\} = Gal(K/L)$

First isom. thm $\Rightarrow G/L' \simeq \operatorname{im}\varphi$ so suffice show φ surj. or equiv. $|\operatorname{im}\varphi| = |Gal(L/F)|$. But

$$|\operatorname{im}\varphi| = |G/L'| = \frac{|G|}{|L'|} = \frac{[K:F]}{[K:L]} = [L:F]$$
$$= |Gal(L/F)| \qquad \square$$

$\boxed{\text{Examples}}$ Eg. 1 Recall $K = \mathbb{Q}(\sqrt[3]{2}, \omega = e^{2\pi i/3})$ is Galois $/\mathbb{Q}$ with Galois group $G = S_3$ on identifying field autom. with permutation of roots $\sqrt[3]{2}, \sqrt[3]{2}\omega, \sqrt[3]{2}\omega^2$.

Below we list intermediate fields & Galois groups of all Galois ext^{n}s (save $K/\mathbb{Q}$).

$$\begin{tikzcd}
 & & K \arrow[dll, dash, "\langle(123)\rangle"'] \arrow[d, dash, "\langle(23)\rangle"] \arrow[dr, dash, "\langle(13)\rangle"] \arrow[drr, dash, "\langle(12)\rangle"] & & \\
\mathbb{Q}(\omega) \arrow[drr, dash, "S_3/\langle(123)\rangle"'] & & \mathbb{Q}(\sqrt[3]{2}) \arrow[d, dash] & \mathbb{Q}(\sqrt[3]{2}\omega) \arrow[dl, dash] & \mathbb{Q}(\sqrt[3]{2}\omega^2) \arrow[dll, dash] \\
 & & \mathbb{Q} & &
\end{tikzcd}$$

G fixes

G permutes

Corresp. subgroups

$$\begin{tikzcd}
 & & 1 \arrow[dll, dash] \arrow[d, dash] \arrow[dr, dash] \arrow[drr, dash] & & \\
\langle(1\,2\,3)\rangle \arrow[drr, dash] & & \langle(23)\rangle \arrow[d, dash] & \langle(13)\rangle \arrow[dl, dash] & \langle(12)\rangle \arrow[dll, dash] \\
 & & G & &
\end{tikzcd}$$

Normal subgroup

conjugate subgroups

Eg.2 $\mathrm{Gal}(\mathbb{Q}(\zeta_5)/\mathbb{Q}) = \langle \sigma : \zeta_5 \mapsto \zeta_5^2 \rangle \simeq \mathbb{Z}/4\mathbb{Z}$.

$\zeta_5 = e^{2\pi i/5}$

$$\begin{tikzcd}
\mathbb{Q}(\zeta_5) \arrow[d, dash, "\text{Gal. group } \langle\sigma^2\rangle \simeq \mathbb{Z}/2\mathbb{Z}"] \\
\mathbb{Q}(\zeta_5 + \zeta_5^{-1} = 2\cos\frac{2\pi}{5}) \arrow[d, dash, "\text{Gal. group } \langle\sigma\rangle/\langle\sigma^2\rangle \simeq \mathbb{Z}/2\mathbb{Z}"] \\
\mathbb{Q}
\end{tikzcd}$$

$\sigma\langle\sigma^2\rangle \in \mathrm{Gal}(\mathbb{Q}(\zeta_5+\zeta_5^{-1})/\mathbb{Q})$ maps

$\zeta_5 + \zeta_5^{-1} \longmapsto \sigma(\zeta_5+\zeta_5^{-1}) = \zeta_5^2 + \zeta_5^{-2} = 2\cos\frac{4\pi}{5}$.