

Lecture 10: Galois action on Galois Correspondence

Aim Lecture: Extract more info from Galois corresp. by studying action of Galois group on it.

Galois Action Fix following notⁿ for this lecture

$K/F =$ finite Galois extⁿ

$G =$ Galois group.

Have Galois corresp

$$L \longmapsto L' = \text{Gal}(K/L)$$

$$\mathcal{F} := \left\{ \begin{array}{l} \text{intermediate} \\ \text{fields of } K/F \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{subgroups} \\ \text{of } G \end{array} \right\} = \mathcal{G}$$

$$H' = K^H \longleftarrow H$$

G acts on \mathcal{F} by: $\sigma \in G, L \in \mathcal{F} \Rightarrow \sigma(L) \in \mathcal{F}$ so σ permutes interm. fields.

Also G acts on \mathcal{G} by conjugⁿ $\sigma: H \mapsto \sigma H \sigma^{-1}$.

Propⁿ 1: $\sigma(L)' = \sigma L' \sigma^{-1}$

Proof: (\supseteq) $\sigma L' \sigma^{-1}$ fixes $\sigma(L)$ \therefore given

$\tau \in L' = \text{Gal}(K/L), \alpha \in L:$

$$\sigma \tau \sigma^{-1}(\sigma(\alpha)) = \underbrace{\sigma \tau(\alpha)}_{\alpha} = \sigma(\alpha) \quad \leftarrow \because \tau \in L'$$

(\subseteq) Apply (\supseteq) to $\sigma(L)$ & σ^{-1} to see $L' \supseteq \sigma^{-1} \sigma(L)' \sigma \Rightarrow \sigma L' \sigma^{-1} \supseteq \sigma(L)'$.

□

Cor: $H \leq G$ is normal iff
 $\sigma(K^H) = K^H$ for all $\sigma \in G$

Proof: Propⁿ \Rightarrow fixed points of G -action on G correspond to fixed points on F .

Normality

Lemma: Let L be an intermediate field of K/F s.t. $\sigma(L) = L$ for all $\sigma \in G$. Then L/F is Galois.

Proof: Let $\alpha_1, \dots, \alpha_n$ be a set of gen. for L/F e.g. an F -basis. Let p_1, \dots, p_n be their min. poly. / F .

Lect. 2 Thm \Rightarrow each p_i factorises $p_i(x) = (x - \beta_1) \dots (x - \beta_r)$ where β_1, \dots, β_r is the G -orbit α_i . Hence $p(x) = p_1(x)p_2(x)\dots p_n(x)$ is sep. Also $\sigma(L) = L \Rightarrow$ all $\beta_j \in L$ so L is the splitting field of $p(x)/F$. \square

Addendum to Fundamental Thm

Let K/F be Galois, $G = \text{Gal}(K/F)$ & $L =$ interm. field.

(a) L/F is Galois iff $L' = \text{Gal}(K/L) \trianglelefteq G$.

(b) In this case $\text{Gal}(L/F) \cong G/L'$
 $G/L' = \frac{\text{Gal}(K/F)}{\text{Gal}(K/L)}$.

Proof: (a) Lemma + Cor. give (\Leftarrow) so we prove (\Rightarrow). $L =$ splitting field of sep. $f(x) \in F[x]$ so $L = F(\alpha_1, \dots, \alpha_n)$ where $\alpha_1, \dots, \alpha_n$ are the roots of $f(x)$.

Any $\sigma \in G$ permutes the α_i so $\sigma(L) = L$.
Cor. + Galois corresp. $\Rightarrow L' \trianglelefteq G$.

(b) If $\sigma \in G$, then (or \Rightarrow) the restriction $\sigma|_L$ of σ to L has image L so gives an isom. $\sigma|_L: L \xrightarrow{\sim} L$.

We thus obtain a group hom.

$$\varphi: \text{Gal}(K/F) \rightarrow \text{Gal}(L/F)$$

$$\sigma \longmapsto \sigma|_L$$

Note $\ker \varphi = \{ \sigma \in G \mid \underbrace{\sigma|_L = \text{id}}_{\text{means } \sigma \text{ fixes } L} \} = \text{Gal}(K/L)$

First isom. thm $\Rightarrow G/L' \cong \text{im } \varphi$ so suffice show φ surj. or equiv.

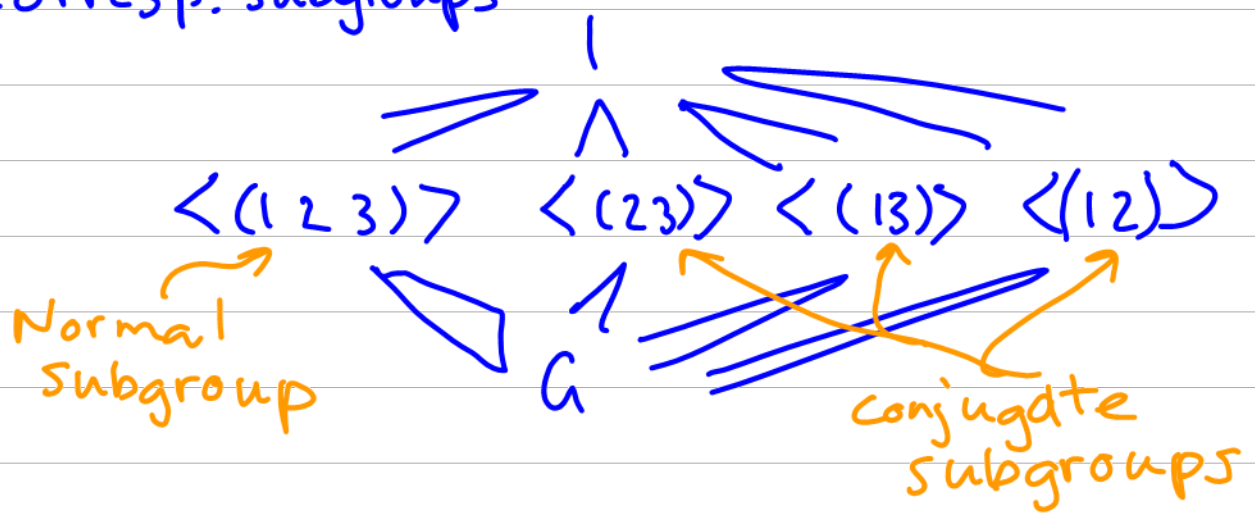
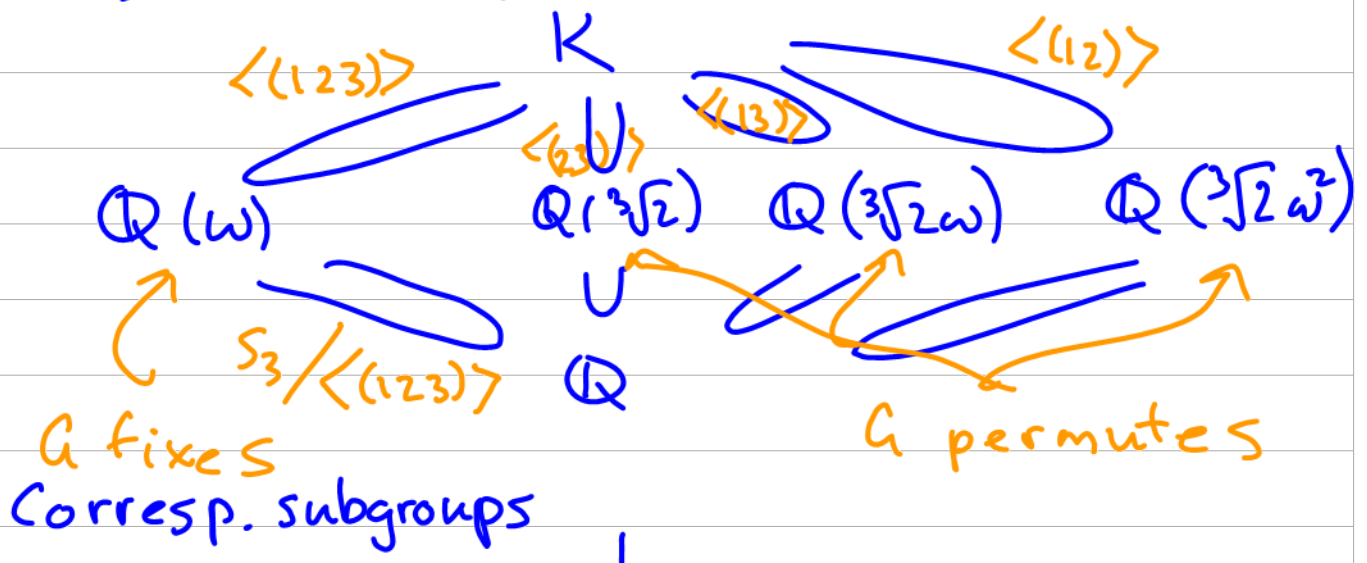
$$|\text{im } \varphi| = |\text{Gal}(L/F)|. \quad \text{But}$$

$$|\text{im } \varphi| = |G/L'| = \frac{|G|}{|L'|} = \frac{[K:F]}{[K:L]} = [L:F]$$

$$= |\text{Gal}(L/F)| \quad \square$$

Examples Eg. 1 recall $K = \mathbb{Q}(\sqrt[3]{2}, \omega = e^{2\pi i/3})$ is Galois / \mathbb{Q} with Galois group $G = S_3$ on identifying field autom. with permutation of roots $\sqrt[3]{2}, \sqrt[3]{2}\omega, \sqrt[3]{2}\omega^2$.

Below we list intermediate fields & Galois groups of all Galois extⁿs (save K/\mathbb{Q}).



Eg. 2 $\text{Gal}(\mathbb{Q}(\zeta_5)/\mathbb{Q}) = \langle \sigma: \zeta_5 \mapsto \zeta_5^2 \rangle \cong \mathbb{Z}/4\mathbb{Z}$.

