Lecture: Ramification 2: Ramification Theory

Aim Lecture: Intro. ramification which gives more info about field extⁿs.

Existence of Models

Defⁿ 1: A (dim 1) function field is a field extⁿ K of $\mathbb{C}$, isom/$\mathbb{C}$ to a finite extⁿ of $\mathbb{C}(t)$

Eg. 1. $\mathbb{C}(X)$ is a function field for any compact Riemann surf. X $\because$ any $f \in \mathbb{C}(X)$ non-constant defines map $f: X \to \mathbb{P}^1$ & Thm last lecture $\Rightarrow$ $f^*: \mathbb{C}(\mathbb{P}^1) \hookrightarrow \mathbb{C}(X)$ finite field extⁿ.

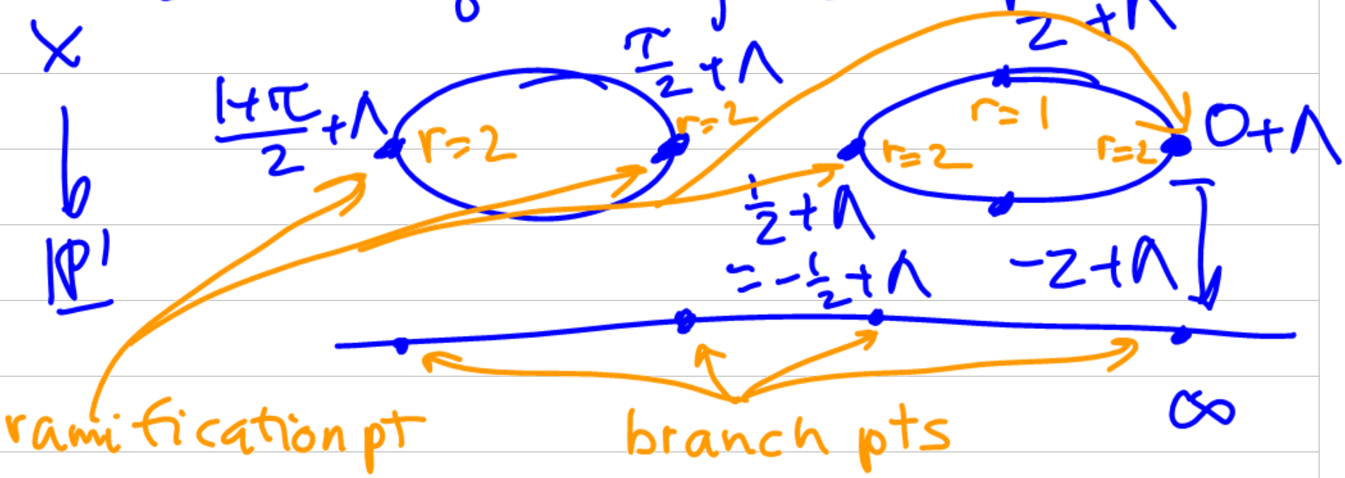
Thm: Any dim 1 function field K is isomorphic /$\mathbb{C}$ to $\mathbb{C}(X)$ for some compact Riemann surface.

Upshot: Together with thm last lecture see any field extⁿ K/F of fn fields can be viewed as φ^* for some hol. map $\varphi: X \to Y$ of compact Riemann surfaces. Geometry of φ gives info about K/F!

Ramification Theory

Eg. 2. Elliptic curve $X = \mathbb{C}/\Lambda$, $\Lambda = \mathbb{Z} + \mathbb{Z}\tau$

Weierstrass $\wp$ fn gives map



More gen. consider hol. map $\varphi: X \to Y$ of compact Riemann surf. Study locally at $x \in X$ & $y = \varphi(x) \in Y$. W.r.t. appropriate co-ord. charts, φ given locally at x by hol.

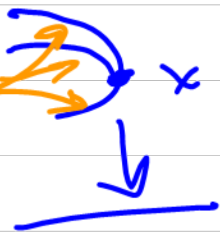
$$f(z) = a_r z^r + a_{r+1} z^{r+1} + \dots$$

for some $a_r \neq 0$, $r = r_x \geq 1$

r=1: φ is loc. an isom. at x.

Def^n 2: If $r_x > 1$ we say φ is ramified at x and over y and r_x is the ramification index.

Changing co-ord., φ loc. of form $w \mapsto w^{r_x}$ so r_x sheets come together



Fact: If $n = [\mathbb{C}(X) : \mathbb{C}(Y)] =: \deg X/Y$, then solving $\varphi(x) = y$ for x amounts to solving deg n poly. i.e. $\varphi^{-1}(y)$ usually has n roots unless there are multiple roots. That's when ramif. occurs.

Galois Case

X = compact Riem. surface.

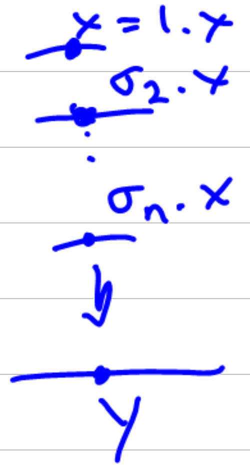
G = finite subgroup of $\mathrm{Gal}(\mathbb{C}(X)/\mathbb{C})$ of order n so functoriality $\Rightarrow$ G acts on X too.

Thm: exist. of models $\Rightarrow$ $\mathbb{C}(X)^G = \mathbb{C}(Y)$ for some compact Riem. surface Y. In fact $Y = {}_G\backslash X$, the set of G-orbits & $\mathbb{C}(Y) \hookrightarrow \mathbb{C}(X)$ corresponds to

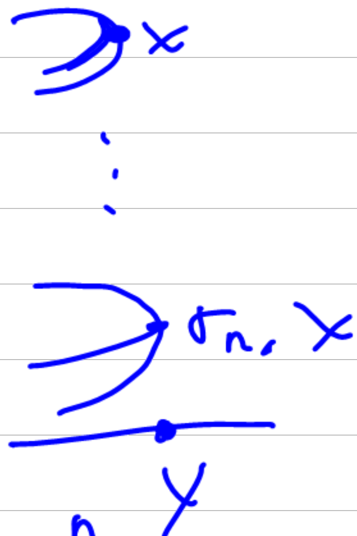
$$\pi : X \to {}_G\backslash X, \quad x \mapsto G.x$$

Ramification: $y = \pi(x)$

Unramified over y



Ramified over y



$|G.x| = n = |G|$ $\qquad$ $|G.x| = \frac{n}{r_x} < n$

$r_y := r_x = \dots = r_{\sigma_n . x} = |\mathrm{Stab}_G\, x|$

Riemann-Hurwitz

Recall Euler characteristic $e(X) = V - E + F$

Thm: (Riemann-Hurwitz) In Galois case above

$$(*) \quad \ldots\ldots \quad \frac{1}{n} e(X) = e(Y) - \sum_{y \in Y} \left(1 - \frac{1}{r_y}\right)$$

Proof Sketch: Triangulate Y fine enough so so all branch pts are vertices. Use inverse image triangulation on X. $\square$

E.g. Suppose $X = \mathbb{P}^1$ so $e(X) = 2$ & RHS$(*) > 0$.

$$\Rightarrow \quad e(Y) = 2 \ \& \ 2 > \sum_y \left(1 - \frac{1}{r_y}\right) \ldots (+)$$

Fact: $e(Y) = 2 \Rightarrow Y \simeq \mathbb{P}^1$ (this is a special case of Lüroth's thm).

Analyze Ram.: y a branch pt $\Rightarrow 1 - \frac{1}{r_y} \geq \frac{1}{2}$

so $(+) \Rightarrow b =$ no. branch pts ≤ 3.

& $(*) \Rightarrow b > 1$ if $G \neq 1$.

b=3: $(+) \Leftrightarrow \frac{1}{r_{y_1}} + \frac{1}{r_{y_2}} + \frac{1}{r_{y_3}} > 1$

Sol$^{\underline{n}}$s are by def$^{\underline{n}}$ Platonic triples

$$(r_{y_1}, r_{y_2}, r_{y_3}) = \begin{cases} (2, 2, \frac{n}{2}) & G \simeq D_{n/2} \\ (2, 3, 3) & G \simeq A_4 \\ (2, 3, 4) & G \simeq S_4 \\ (2, 3, 5) & G \simeq A_5 \end{cases}$$

can assume $r_{y_1} \leq r_{y_2} \leq r_{y_3}$

N.B $(*)$ gives $|G| = n$ & $b = 2 \Rightarrow G$ cyclic.