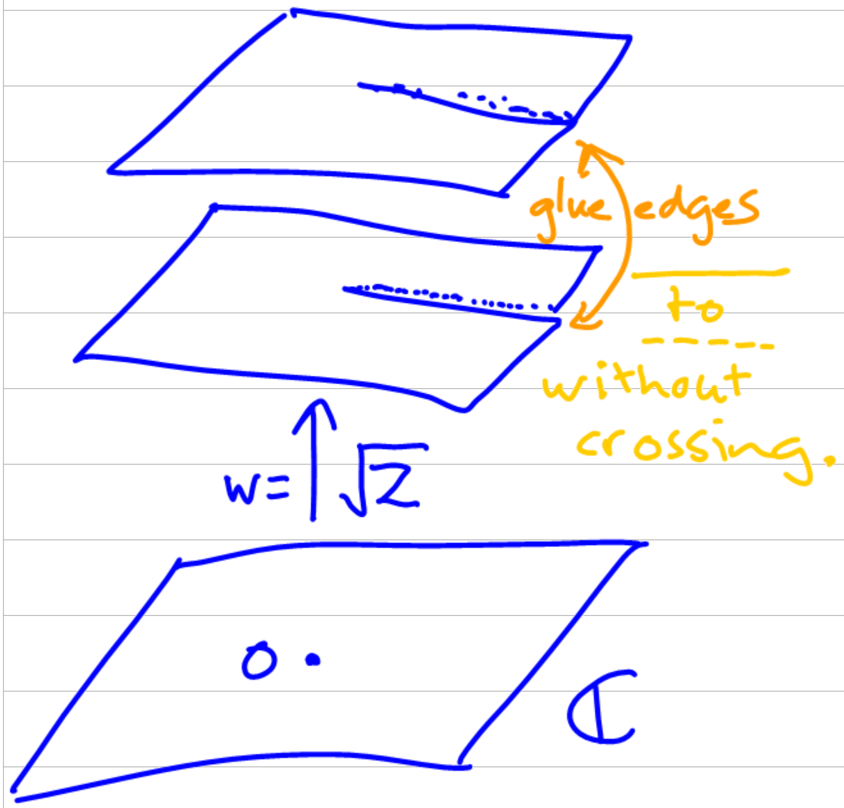
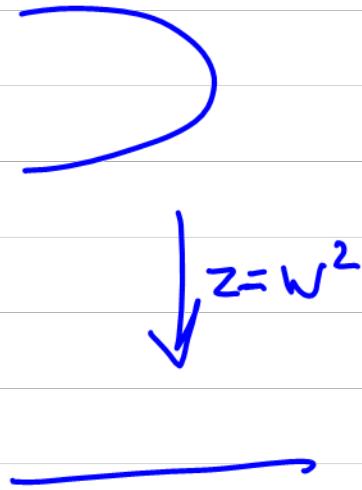
Lecture: Ramification 1: Riemann Surfaces

Aim Lecture: Intro. Riemann surfaces & assoc. field ext^{n}s.

Riemann adjoins $\sqrt{\ }$



Draw $\mathbb{R}$-locus inverse map



Riemann Surface

Def^n 1: A compact Riemann surface X, is a compact 2-dim manifold whose charts have form

$$\varphi_\alpha : \underset{\substack{\text{in}\\ \mathbb{C}}}{U_\alpha} \longrightarrow X$$

& transition fns $\varphi_\alpha^{-1}\varphi_\beta$ are holomorphic on domain of def^n. N.B. X is oriented!

E.g. 1. Riemann sphere

$$\mathbb{P}^1_{\mathbb{C}} = \mathbb{C} \,\dot{\cup}\, \{\infty\}$$

Charts $\varphi_\infty : \mathbb{C} \longrightarrow \mathbb{P}^1_{\mathbb{C}} - \infty$

$z \longmapsto z$

$\varphi_0 : \mathbb{C} \longrightarrow \mathbb{P}^1_{\mathbb{C}} - 0$

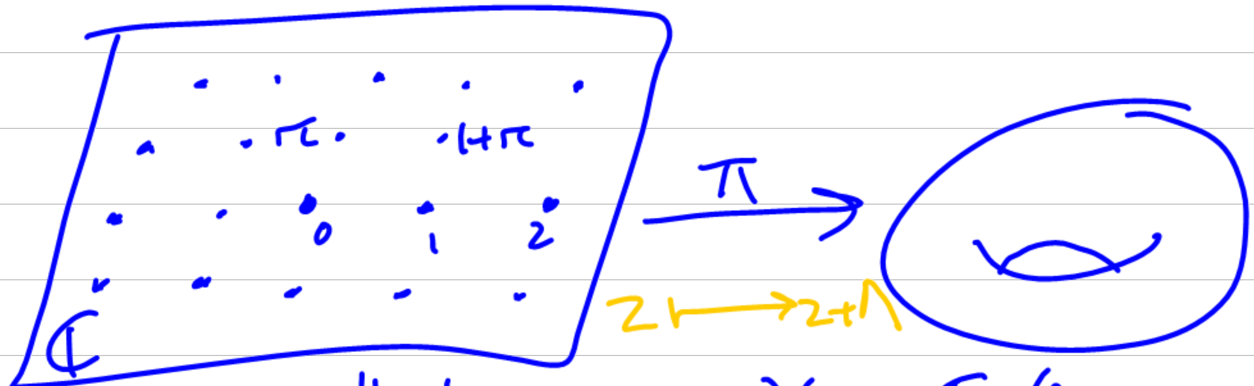
$y \longmapsto y^{-1}$

$\varphi_\infty^{-1}\varphi_0 = y^{-1}$

E.g. 2. Elliptic curve.

Pick $\tau \in \mathbb{C}$ with $\operatorname{im}\tau > 0$.

Consider lattice $\Lambda = \mathbb{Z} + \mathbb{Z}\tau < \mathbb{C}$



elliptic curve $X = \mathbb{C}/\Lambda$ is a compact Riemann surface with charts $\pi|_U : U \to X$ for suff. small $U < \mathbb{C}$.

Field of Meromorphic Functions

Let X = compact Riem. surface.

Def^{n} 2: $\mathbb{C}(X)$ is the field of meromorphic functions on X i.e. functions f holomorphic on X except for a finite no. poles. Equiv. f is a holomorphic fn $X \to \mathbb{P}^1_{\mathbb{C}}$.

Eg. 1 again $\mathbb{C}(\mathbb{P}^1_{\mathbb{C}}) = \mathbb{C}(t)$.

Eg. 2 again. $X = \mathbb{C}/\Lambda$, $\Lambda = \mathbb{Z} + \mathbb{Z}\tau$.

Mero. fns f on X correspond to doubly periodic mero. fns on $\mathbb{C}$ i.e.

$$f(z+1) = f(z), \quad f(z+\tau) = f(z).$$

Prop$^{\underline{n}}$-Def$^{\underline{n}}$: The Weierstrass $\wp$-function

$$\wp(z) = \frac{1}{z^2} + \sum_{\lambda \in \Lambda - 0} \frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2}$$

is meromorphic. Furthermore

$$\wp'(z)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3$$

for some $g_2, g_3 \in \mathbb{C}$.

Fact: $\mathbb{C}(X) = \mathbb{C}(\underset{t}{\underset{\shortparallel}{\wp(z)}}, \wp'(z)) \simeq \mathbb{C}(t, \sqrt{4t^3 - g_2 t - g_3})$

$\boxed{\text{Functoriality}}$

Let $\varphi : X \to Y$ be a nonconstant holomorphic map of compact Riem. surfaces so φ is surj. by Liouville's thm.

If $f: Y \to \mathbb{P}^1_{\mathbb{C}}$ holomorphic, so is $\varphi^* f := f \circ \varphi : X \to \mathbb{P}^1$

Thm: (a) φ induces a field hom

$$\varphi^* : \mathbb{C}(Y) \longrightarrow \mathbb{C}(X).$$

(b) φ is generically $d:1$ where

$$d = [\mathbb{C}(X) : \mathbb{C}(Y)] < \infty$$

(c) If $\psi: Y \to Z$ hol. non-const. then

$$(\psi\varphi)^* = \varphi^* \psi^*.$$

(d) Any field hom. $\iota : \mathbb{C}(Y) \to \mathbb{C}(X)$ over $\mathbb{C}$ has form ρ^* for some hol. $\rho : X \to Y$.
No Proof.

Eg. 1&2. Weierstrass $\wp$ fn induces

$$\varphi : X = \mathbb{C}/\Lambda \longrightarrow \mathbb{P}^1_{\mathbb{C}}.$$
$$z + \Lambda \longmapsto \wp(z)$$

Recall $\mathbb{C}(\mathbb{P}^1_{\mathbb{C}}) = \mathbb{C}(t)$ where $t \leftrightarrow id : \mathbb{P}^1_{\mathbb{C}} \to \mathbb{P}^1_{\mathbb{C}}$.

$$\begin{tikzcd}
X \arrow[r, "\varphi \leftrightarrow \wp"] \arrow[dr, "\varphi^* t \leftrightarrow \varphi"'] & \mathbb{P}^1_{\mathbb{C}} \arrow[d, "id \leftrightarrow t"] \\
& \mathbb{P}^1_{\mathbb{C}}
\end{tikzcd}$$

$$\varphi^* : \mathbb{C}(t) \to \mathbb{C}(\wp(z), \wp'(z))$$
$$t \longmapsto \wp(z)$$

$\therefore$ $\varphi : X \to \mathbb{P}^1_{\mathbb{C}}$ is gen. 2:1.
$\wp$ even $\Rightarrow$ $\wp(\pm z) = \wp(z)$ & $\wp$ has double pole at 0.

Have comm. diagrams

$$\begin{tikzcd}
X \arrow[rr, "\sigma"] \arrow[dr, "\varphi"'] & & X \arrow[dl, "\varphi"] \\
& \mathbb{P}^1_{\mathbb{C}} &
\end{tikzcd}
\qquad z+\Lambda \longmapsto -z+\Lambda
\qquad
\begin{tikzcd}
\mathbb{C}(X) & & \mathbb{C}(X) \arrow[ll, "\sigma^*"'] \\
& \mathbb{C}(\mathbb{P}^1) = \mathbb{C}(t) \arrow[ul, "\varphi^*"] \arrow[ur, "\varphi^*"'] &
\end{tikzcd}$$

$\varphi\sigma = \varphi$ $\qquad$ $\sigma^*\varphi^* = \varphi^*$

$\Rightarrow$ $\sigma^* \in Gal(\mathbb{C}(X)/\mathbb{C}(\mathbb{P}^1))$

$$\sigma^* \wp(z) = \wp(-z) = \wp(z)$$
$$\sigma^* \wp'(z) = \wp'(-z) = -\wp'(z).$$

σ^* gen. Galois group.