Infinite 3: Infinite Galois Correspondence

Aim Lecture: Give Galois correspondence in infinite case.

Galois Correspondence

Let K/F = Galois extⁿ. Compactness of Galois groups suggests

Lemma: Let L = intermediate field. Then $H = Gal(K/L)$ is a closed subgroup of $G = Gal(K/F)$.

Proof: Case L/F finite: Let $\tilde{L}$ = Galois closure of L/F so is a finite Galois extⁿ. Consider restriction map

$$\pi : G = Gal(K/F) \longrightarrow Gal(\tilde{L}/F) \supseteq Gal(\tilde{L}/L)$$

which is cont. $H_1 := \pi^{-1}(Gal(\tilde{L}/L))$ is closed $\because$ top. on $Gal(\tilde{L}/F)$ is discrete.

To complete L/F finite case, note $H = H_1$. Indeed $\sigma \in G$ fixes L iff $\pi(\sigma) = \sigma|_{\tilde{L}}$ fixes L.

Case L/F infinite: $L = \bigcup L_\alpha$ where L_α/F are finite extⁿs. Then
$$Gal(K/L) = \bigcap Gal(K/L_\alpha)$$
is closed being the intersection of such. $\square$

Rem: The profinite topology on $\mathrm{Gal}(K/L)$ is the same as the subspace topology induced by inclusion $\mathrm{Gal}(K/L) \hookrightarrow \mathrm{Gal}(K/F)$.

Thm: Let K/F = Galois ext$^{\underline{n}}$, Galois group G. The Galois correspondence restricts to inverse bijections:

$$L \longmapsto L' = \mathrm{Gal}(K/L)$$

$$\left\{\begin{matrix}\text{Intermediate} \\ \text{fields of } K/F\end{matrix}\right\} \rightleftarrows \left\{\begin{matrix}\text{closed subgroups} \\ \text{of } G\end{matrix}\right\}$$

$$H' = K^H \longleftarrow\!\!\!\shortmid H$$

Proof: Check $L = L'' = K^{\mathrm{Gal}(K/L)}$. Note $L \subseteq L''$. Suppose $\alpha \in K - L$ & let $\tilde{L}$ be a Galois closure of $L(\alpha)/L$. Finite Galois theory $\Rightarrow \exists\, \tilde{\sigma} \in \mathrm{Gal}(\tilde{L}/L)$ with $\tilde{\sigma}(\alpha) \neq \alpha$. Lecture: Infinite 2 lemma $\Rightarrow$ can lift $\tilde{\sigma}$ to $\sigma \in \mathrm{Gal}(K/L)$. Then $\sigma(\alpha) \neq \alpha \Rightarrow \alpha \notin L''$ so $L = L''$.

Check $H = H'' = \mathrm{Gal}(K/K^H)$. Note $H \subseteq H''$. Suppose to contrary $\sigma \in H'' - H$. H closed $\Rightarrow \exists$ open nbhd of σ of form

$$U = \{\tau \in G \mid \tau|_{K_i} = \sigma|_{K_i}, \, i=1,\dots,n\}$$

where K_i/F are finite Galois s.t.

$H \cap U \overset{*}{=} \varnothing$.

Suppose K_i = split. field of $f_i(x)/F$ & let L = split. field of $f_1 f_2 \dots f_n / K^H$.

Consider restriction hom

$$\pi : H'' = Gal(K/K^H) \longrightarrow Gal(L/K^H)$$

Note $L^{\pi(H)} = K^H$ so finite Galois corresp.

$\Longrightarrow \pi(H) = Gal(L/K^H) \ni \sigma|_L$.

$\therefore \exists \, \tau \in H$ with $\pi(\tau) = \tau|_L = \sigma|_L$.

But $L \supseteq K_i$ so $\tau \in U \cap H$ contradicting $*$.

$\square$.

Example

Fix prime q. Recall the $\mathbb{F}_{q^n} \subset \overline{\mathbb{F}_q}$ ordered by $\mathbb{F}_{q^n} \leq \mathbb{F}_{q^m} \iff n|m$. For prime p, $\mathbb{F}_{q^{p^\infty}} := \bigcup_n \mathbb{F}_{q^{p^n}}$ is an intermediate field of $\overline{\mathbb{F}_q}/\mathbb{F}_q$.

$\mathbb{F}_{q^{p^\infty}}/\mathbb{F}_q$ is Galois with Galois group inverse limit of

$$\cdots \to \mathrm{Gal}(\mathbb{F}_{q^{p^3}}/\mathbb{F}_q) \to \mathrm{Gal}(\mathbb{F}_{q^{p^2}}/\mathbb{F}_q) \to \mathrm{Gal}(\mathbb{F}_{q^p}/\mathbb{F}_q)$$

$$\cdots \to \mathbb{Z}/p^3\mathbb{Z} \to \mathbb{Z}/p^2\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z}$$

denoted $\hat{\mathbb{Z}}_p := \varprojlim_n \mathbb{Z}/p^n\mathbb{Z}$ & called the p-adic integers. Elts best represented as power series in p:

$$\underbrace{a_0}_{\text{image in } \mathbb{Z}/p\mathbb{Z}} + a_1 p + a_2 p^2 + \cdots, \quad a_i \in \{0, \dots, p-1\}$$

image in $\mathbb{Z}/p^2\mathbb{Z}$ etc.

In particular $\hat{\mathbb{Z}}_p$ is uncountable.

Proj^n – $\pi_p : \hat{\mathbb{Z}} \to \hat{\mathbb{Z}}_p$: Composite$_n$

$$\hat{\mathbb{Z}} \hookrightarrow \prod_m \mathbb{Z}/m\mathbb{Z} \xrightarrow[\substack{m=p^n\\ \text{component}}]{\text{proj}^n\text{ onto}} \mathbb{Z}/p^n\mathbb{Z}$$

defines via univ. property for inverse limit $\hat{\mathbb{Z}}_p$ an hom. $\pi_p : \hat{\mathbb{Z}} \to \hat{\mathbb{Z}}_p$.

Facts: (a) These induce an isom $\hat{\mathbb{Z}} \simeq \prod_{p \text{ prime}} \hat{\mathbb{Z}}_p$. via Chinese Remainder Thm.

(b) The subgroup of $\hat{\mathbb{Z}}$ corresp to $\mathbb{F}_{q^{p^\infty}}$ is $\ker \pi_p$.

Proof: ex.

Fact (a) $\Rightarrow$ $\mathbb{Z} = \langle 1 \rangle \subsetneq \hat{\mathbb{Z}}$ so it can't be closed as $\overline{\mathbb{F}_q}^{\mathbb{Z}} = \mathbb{F}_q$. We see closure of $\mathbb{Z}$ in $\hat{\mathbb{Z}}$ is $\hat{\mathbb{Z}}$ so we often say the Frob. hom generates $\mathrm{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q)$ topologically.