Infinite 2: Infinite Galois Groups

Aim Lecture: View Galois groups of infinite Galois ext$^{\underline{n}}$s as profinite groups.

Infinite Galois Ext$^{\underline{n}}$s

Def$^{\underline{n}}$ 1: A (not nec. finite) field ext$^{\underline{n}}$ K/F is Galois if it is the union of finite Galois ext$^{\underline{n}}$s i.e. (ex) K is obtained from F by adjoining "all" the roots of a set $\{f_\alpha\}$ of sep. poly $f_\alpha \in F[x]$ over F.

N.B. K/F Galois $\Rightarrow$ K/F algebraic still.

Many results about finite Galois ext$^{\underline{n}}$s hold in infinite case with same proof except Zorn's lemma is used instead of induction. E.g.

Lemma: Let K/F = Galois ext$^{\underline{n}}$ with Galois group G. Let L = intermed. field s.t. L/F is Galois. Then restriction

$$\begin{aligned} \rho: G = Gal(K/F) &\longrightarrow Gal(L/F) \\ \sigma &\longmapsto \sigma|_L \end{aligned}$$

is a surj. group hom with kernel $L' = Gal(K/L)$ so $Gal(L/F) \simeq G/L'$.

Inverse System of Finite Galois Groups

Let K/F = Galois ext$^{\underline{n}}$, Galois group G.

Step 1: Let $A = \{K_\alpha \subset K \text{ subfield} \mid K_\alpha/F \text{ is finite Galois}\}$.

Partially order A by inclusion.

<u>Fact</u>: $\{K_\alpha, K_\alpha \overset{\varphi_{\alpha\beta}}{\hookrightarrow} K_\gamma\}$ directed system of fields

<u>Why?</u>

$\alpha, \beta \in A$.

$K_\alpha = F(a)$, $K_\beta = F(b)$.

$$\begin{tikzcd} & K_\alpha \arrow[dr] & \\ F \arrow[ur, hook] \arrow[dr] \arrow[rr, bend right=60, "\text{inclusion}"'] & & K_\gamma \\ & K_\beta \arrow[ur, hook] & \end{tikzcd}$$

$\exists ?\ K_\gamma$

K_γ = Galois closure of $F(a,b)$ is Galois $/F$ & contains K_α & K_β.

Also composite of field inclusions is inclusion

<u>Step 2:</u> Reversing order on A get inverse system

$\{G_\alpha = Gal(K_\alpha/F) \simeq G/K_\alpha'\}$ with group hom

$K_\alpha \subseteq K_\gamma \rightsquigarrow$ restriction hom

$$\varphi_{\gamma\alpha} : G_\gamma \longrightarrow G_\alpha : \sigma \mapsto \sigma|_{K_\alpha}$$

Why? Composite of restrictions is restriction & Fact above.

Infinite Galois Groups

<u>Thm</u>: Let K/F be a Galois extⁿ with Galois group G & $\{G_\alpha = Gal(K_\alpha/F)\}_{\alpha \in A}$ the inverse system of Galois groups in Step 2 above. Then $G \simeq \varprojlim G_\alpha$ so is profinite.

Proof: Lemma $\Rightarrow$ we have restriction hom. $\psi_\alpha : G \longrightarrow G_\alpha$.

If $K_\alpha \subseteq K_\beta$

$$\begin{tikzcd}
\mathrm{Gal}(K/F) \arrow[d, "\psi_\beta"'] \arrow[dr, "\psi_\alpha"] & \\
\mathrm{Gal}(K_\beta/F) \arrow[r, "\varphi_{\beta\alpha}"] & \mathrm{Gal}(K_\alpha/F)
\end{tikzcd}$$

$$\psi_\alpha = \varphi_{\beta\alpha}\psi_\beta .$$

Univ. property in Lecture: Infinite1 $\Rightarrow \psi_\alpha$ induce group hom $\psi : G \to \varprojlim G_\alpha$

<u>ψ inj.?</u> Suppose $1 = \psi(\sigma)$ so for every $\alpha \in A$, $1 = \psi_\alpha(\sigma) = \sigma|_{K_\alpha}$. But $K = \bigcup K_\alpha$ so $\sigma = id_K$ & ψ is inj.

<u>ψ surj?</u> Let $(\sigma_\alpha) \in \varprojlim G_\alpha$. We seek to define $\sigma \in G$ with $\psi(\sigma) = (\sigma_\alpha)$ i.e. $\sigma|_{K_\alpha} = \sigma_\alpha$ for all α.

Let $\varepsilon \in K$ so $\varepsilon \in K_\beta$ for some K_β/F finite Galois. Define $\sigma(\varepsilon) = \sigma_\beta(\varepsilon)$. This is independent of choice K_β for if $\varepsilon \in K_\alpha$ too, $\exists\, K_\gamma/F$ finite Galois containing K_α & K_β. Now σ_γ restricts to σ_α & σ_β so

$\sigma_\alpha(\varepsilon) = \sigma_\gamma(\varepsilon) = \sigma_\beta(\varepsilon)$.

Finally, $\sigma: K \to K$ is a field autom. $/F$ $\therefore$ each σ_α is (ex. check).

$\boxed{\text{Absolute Galois Group}}$

Def^n 2: Let F = field & F^{sep} = the sep. closure of F. Note F^{sep}/F is Galois. The absolute Galois group of F is the profinite group $Gal(F^{sep}/F)$.

E.g. For p prime, $G := Gal(\mathbb{F}_{p_{\shortparallel}}^{sep}/\mathbb{F}_p) \simeq \hat{\mathbb{Z}}$
$\overline{\mathbb{F}_p}$.

Why? $G = \varprojlim G_n$
where $G_n = Gal(\mathbb{F}_{p^n}/\mathbb{F}) \simeq \mathbb{Z}/n\mathbb{Z}$.
gen. by Frobenius. Furthermore, Galois corresp. on $\mathbb{F}_{p^n}$ shows $\mathbb{F}_{p^n} \supseteq \mathbb{F}_{p^m} \iff m|n$ in which case restriction map is

$$\begin{array}{ccc} Gal(\mathbb{F}_{p^n}/\mathbb{F}_p) & \to & Gal(\mathbb{F}_{p^m}/\mathbb{F}_p) \\ \wr\!\parallel & & \\ \mathbb{Z}/n\mathbb{Z} & \longrightarrow & \mathbb{Z}/m\mathbb{Z}. \\ a+n\mathbb{Z} & \longmapsto & a+m\mathbb{Z} \end{array}$$

since Frob. restricts to Frobenius.