Infinite Galois I: Pro-finite Groups

Aim Lecture: Infinite Galois groups require extra top. structure introduced today.

Inverse Limits

Let A = set partially ordered by $\lesssim$.
Consider data

ⓐ For each $\alpha \in A$, a group G_α

ⓑ For $\alpha, \beta \in A$ with $\alpha \lesssim \beta$, a group hom
$$\varphi_{\alpha\beta} : G_\alpha \longrightarrow G_\beta \quad \text{s.t.}$$

$$\varphi_{\alpha\gamma} = \varphi_{\beta\gamma}\varphi_{\alpha\beta} : \begin{tikzcd} G_\alpha \arrow[r, "\varphi_{\alpha\beta}"] \arrow[rr, bend right, "\varphi_{\alpha\gamma}"'] & G_\beta \arrow[r, "\varphi_{\beta\gamma}"] & G_\gamma \end{tikzcd}$$

Def^{n} 1: The above data is an inverse (resp. direct) system if for any $\alpha, \beta \in A$ there's some $\gamma \in A$ with $\gamma \lesssim \alpha$, $\gamma \lesssim \beta$ (resp. $\gamma \gtrsim \alpha$, $\gamma \gtrsim \beta$).

Rem: If the groups & hom in ⓐ, ⓑ above are fields & field hom. we may speak of an inverse/direct system of fields etc.

Prop^{n}-Def^{n} 1: Consider an inverse system $\{G_\alpha\}$ of groups as above. Define
$$G = \{ (g_\alpha) \in \prod G_\alpha \mid \varphi_{\alpha\beta}(g_\alpha) = g_\beta \}.$$
Then $G \leq \prod G_\alpha$ called the inverse limit & denoted $\varprojlim G_\alpha$.

It satisfies the following universal property. Given group H & group hom $\psi_\alpha : H \to G_\alpha$ satisfying

$$\begin{tikzcd} H \arrow[d, "\psi_\alpha"'] \arrow[dr, "\psi_\beta"] & \\ G_\alpha \arrow[r, "\varphi_{\alpha\beta}"'] & G_\beta \end{tikzcd} \qquad \psi_\beta = \varphi_{\alpha\beta} \circ \psi_\alpha \quad \text{for all } \alpha \preccurlyeq \beta,$$

$$\psi : H \to \varprojlim G_\alpha$$
$$h \mapsto (\psi_\alpha(h))$$

is a well-defined group hom.

Proof: easy ex. $\square$

Eg. 1 A = set of positive integers ordered by $m \preccurlyeq n$ if $n | m$. Let $G_n = \mathbb{Z}/n\mathbb{Z}$ & $\varphi_{mn} : \mathbb{Z}/m\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$

$$a + m\mathbb{Z} \mapsto a + n\mathbb{Z}$$

whenever $n | m$.

This is inverse system with inverse limit $\varprojlim \mathbb{Z}/n\mathbb{Z} =: \hat{\mathbb{Z}}$.

Consider quotient maps $\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$. Univ. property gives hom $\mathbb{Z} \to \hat{\mathbb{Z}}$.

Topological Groups

Def$^{\underline{n}}$ 2: A topological group is a group G equipped with topology s.t.

ⓐ the mult$^{\underline{n}}$ map $\mu : G \times G \to G$ is cont.

& ⓑ the inverse map $\iota : G \to G : g \mapsto g^{-1}$ is cont.

Eg. 2. $\mathbb{R}$ with Euclid. top. is a topological group.

Prop$^{\underline{n}}$: Let $G =$ top. group.

(a) $U \subseteq G$ open $\Rightarrow$ gU open for any $g \in G$.

(b) Any open subgroup $U \leq G$ is closed.

Proof: (a) Left mult$^{\underline{n}}$ by g is an homeo: it factors as

$$\begin{array}{ccccc} G & \longrightarrow & G \times G & \xrightarrow{\mu} & G \\ h & \longmapsto & (g,h) & \longmapsto & gh \end{array}$$

both cont.

(b) $G - U = \bigcup_{g \notin U} gU$ open by (a). $\square$

Profinite Groups

Let $\{G_\gamma, \varphi_{\alpha\beta}\}$ = inverse system of finite groups. Put discrete top. on each G_γ and product top. on $G = \prod_\gamma G_\gamma$.

Then $\prod G_\gamma$ is a top. group $\because$

$$\begin{tikzcd} G \times G \arrow[r, "\mu"] \arrow[d] & G \arrow[r] & G_\gamma \\ G_\gamma \times G_\gamma \arrow[urr, "\mu_{G_\gamma}"'] & & \end{tikzcd} \qquad \begin{tikzcd} G \arrow[r, "\iota"] \arrow[d] & G \arrow[r] & G_\gamma \\ G_\gamma \arrow[urr, "\iota_{G_\gamma}"'] & & \end{tikzcd}$$

proj$^{\underline{n}}$ map

are continuous.

N.B Topology on G not discrete if index set infinite!!

Prop$^{\underline{n}}$-Def$^{\underline{n}}$ 2: The subspace topology on $\varprojlim G_\gamma$ inherited from $\varprojlim G_\gamma \hookrightarrow G$ makes $\varprojlim G_\gamma$ a top. group. Any such topological group is called a pro-finite group.

Proof: Mult$^{\underline{n}}$ & inverse on G restrict to cont. maps on $\varprojlim G_\gamma$. $\square$

Lemma: With above not$^{\underline{n}}$

ⓐ $\varprojlim G_\gamma \subseteq G$ is closed.

ⓑ $\varprojlim G_\gamma$ is compact.

ⓒ The open subgroups of $\varprojlim G_\alpha$ are the closed subgroups of finite index.

Proof: ⓐ For $\alpha \precsim \beta$, we have cont. map

$$\delta_{\alpha\beta} : \prod_\gamma G_\gamma \to G_\beta : (g_\gamma) \longmapsto \varphi_{\alpha\beta}(g_\alpha) g_\beta^{-1}$$

Then

$$\varprojlim G_\gamma = \bigcap_{\alpha \precsim \beta} \delta_{\alpha\beta}^{-1}(1_{G_\beta}) \text{ is closed}$$

closed $\because$ G_β discrete.

ⓑ $\Leftarrow$ ⓐ + Tychonoff

ⓒ ($\Longrightarrow$) Group $H \leq \varprojlim G_\gamma$ open $\Rightarrow$

closed by Prop$^{\underline{n}}$ ⓑ. Cosets of H are a disjoint open cover of $\varprojlim G_\alpha$ so compactness $\Rightarrow$ H finite index.

($\Longleftarrow$) $H \leq \varprojlim G_\gamma$ closed finite index

$$\Rightarrow \varprojlim G_\gamma - H = \bigcup_{g \notin H} gH \text{ also closed.}$$

$\square$