Graphs 3: Nielsen–Schreier Theorem

Aim Lecture: Use Galois correspondence & graph theory to prove result about groups.

Symmetries of $BF(X)$

Let X = set, $BF(X) = (G, G\times X \cup G\times X^{-1})$

Note G acts freely on $BF(X)$ by left mult$^{\underline{n}}$ i.e. for $\sigma, g \in G$, $x \in X$

$$\sigma . g = \sigma g$$

$$\sigma.(e : g \xrightarrow{x} gx) = \sigma.e : \sigma g \xrightarrow{x} \sigma g x$$

$$\sigma.(\bar{e} : g \xleftarrow{x^{-1}} gx) = \sigma.\bar{e} : \sigma g \xleftarrow{x^{-1}} \sigma g x$$

Hence any subgroup $H \leq G$ also acts freely on $BF(X)$. We get a cover of graphs $p : BF(X) \to H\backslash BF(X) =: \Gamma$.

$\Gamma_0 = H\backslash G$ set of cosets of H in G.

Nielsen-Schreier Thm

Thm (Nielsen-Schreier): Any subgroup H of the free group $F(X)$ is free. Furthermore, if $[F(X):H] = r$, $|X| = n < \infty$ then $H \simeq F(Y)$ where $|Y| = r(n-1)+1$.

Proof: Let $\Gamma = H\backslash BF(X)$. The good conceptual proof uses Galois correspond. to see $H \cong Gal(BF(X)/\Gamma)$ & athm that $Gal(\tilde{\Gamma}/\Gamma)$ is free whenever $\tilde{\Gamma}$ is a tree. We argue directly instead.

Zorn's lemma $\Rightarrow \exists$ maximal tree $T \subseteq \Gamma$ ie. T is a subgraph which is a tree & contains all vertices of Γ.

Let $Y = (\Gamma_1 - T_1) \cap p_1(G \times X)$ so $\Gamma_1 - T_1 = Y \dot{\cup} \bar{Y}$.

Note if $r, n < \infty$

$$|Y| = |\Gamma_1| - (|\Gamma_0| - 1)$$
$$= rn - r + 1 \quad = r(n-1)+1$$

Define $\varphi : H \to F(Y)$: Let $h \in H$.

$BF(X)$ a tree $\Rightarrow \exists$ unique irred. path $\alpha_1 \ldots \alpha_n$ from $1 \rightsquigarrow h$. If we delete all edges in T_1 from $p_1(\alpha_1) \ldots p_1(\alpha_n)$, we get a word $\varphi(h)$ in $Y^{\pm 1}$. N.B. Get same answer if any path (not nec. irred.) $1 \rightsquigarrow h$ used.

Check φ is an hom: Let $h, h' \in H$

$\alpha_1 \ldots \alpha_m$ path $1 \rightsquigarrow h$

$\alpha_1' \ldots \alpha_n'$ " $1 \rightsquigarrow h'$.

path $1 \rightsquigarrow hh'$ is

$\alpha_1 \ldots \alpha_m (h.\alpha_1') \ldots (h.\alpha_n')$.

Since p identifies H-orbits

path $p(\alpha_1) \ldots p(\alpha_m) p(h.\alpha_1') \ldots p(h.\alpha_n')$

$= p(\alpha_1)\dots p(\alpha_m)\,p(\alpha_1')\dots p(\alpha_n')$

$\therefore$ see $\varphi(hh') = \varphi(h)\varphi(h')$.

Check φ surjective: Since $F(Y)$ gen. by Y, suffice check $y \in \operatorname{im}\varphi$ for all $y \in Y$.

Consider paths in T

$t_1 \dots t_m$ from $H \rightsquigarrow s(y)$

$t_1' \dots t_n'$ from $t(y) \rightsquigarrow H$

$\therefore$ Get path in Γ

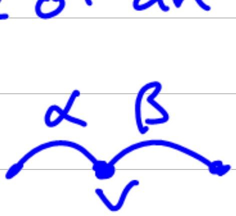
$w_y := t_1 \dots t_m\, y\, t_1' \dots t_n'$ from $H \rightsquigarrow H$.

Lecture: Graphs 2 Prop $\overset{n}{=}1$ $\Rightarrow$ lifts to path $\alpha_1 \dots \alpha_{m+n+1}$ from $1 \rightsquigarrow h \in H$.

Note $\varphi(h) = y$.

Check φ injective: Need

<u>Lemma</u>: For any cover $p : \tilde{\Gamma} \to \Gamma$ of graphs, the image of an irred. path is irred.

Proof. Consider 

in $\tilde{\Gamma}$, $\beta \neq \bar{\alpha}$.

If $p_1(\beta) = \overline{p_1(\alpha)} = p_1(\bar{\alpha})$ then p_1 not injective on edges starting at v

$\square$

Let $h \in H - 1$ with $\varphi(h) = 1$.
Consider irred. path $\alpha_1 \alpha_2 \ldots \alpha_n$ from $1 \rightsquigarrow h$.

If all $p_1(\alpha_i) \in T$, then $p_1(\alpha_1) \ldots p_1(\alpha_n)$ is a path $H \rightsquigarrow H$ which is irred. by the lemma. Now T is a tree so $n=0$ $\Rightarrow h = 1$.

Otherwise, normal form thm on $F(Y)$ $\Rightarrow$ there's a subpath
$p_1(\alpha_i) \ldots p_1(\alpha_j)$
where $\overline{p_1(\alpha_i)} = p_1(\alpha_j) \in Y^{\pm 1}$
& $w := p_1(\alpha_{i+1}) \ldots p_1(\alpha_{j-1})$ lies in T.
Since w is an irred. path from $t(p_1(\alpha_i))$ $\rightsquigarrow t(p_1(\alpha_i))$, we have $j = i+1$ as above, which also contradicts the lemma.

$\square$

Relation to Fundamental Group

Just as for top. spaces you can define the fundamental group $\pi_1(\Gamma)$ of a connected graph Γ using loops. $\exists$ cover $p: \tilde{\Gamma} \to \Gamma$ with $\tilde{\Gamma}$ a tree with $Gal(\tilde{\Gamma}/\Gamma) \cong \pi_1(\Gamma)$. Moreover, p is Galois in sense p isomorphic to $\tilde{\Gamma} \to \pi_1(\Gamma) \backslash \tilde{\Gamma}$ or equiv., $\pi_1(\Gamma)$ acts transitively on the fibres of p.