Graphs 2: Galois Covers

Aim Lecture: Give Galois correspondence for covers of graphs.
(N.B. $\exists$ analogous theory for top. spaces.)

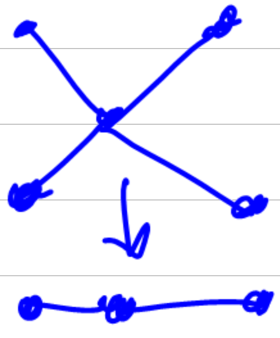
Covers Let Γ = connected graph

Def^{n} 1: A cover of Γ is a morphism of connected graphs $p:\tilde{\Gamma}\to\Gamma$ s.t.

(a) p_0 is surjective &

(b) p is locally bijective i.e. for any $v\in\tilde{\Gamma}_1$ induced map
$$p^v:\{\tilde{e}\mid s(\tilde{e})=v\}\to\{e\mid s(e)=p_0(v)\}$$
$$\tilde{e}\longmapsto p_1(\tilde{e})$$
is bijective.

Eg. 1. Not



but $B\mathbb{Z}$ =

$\cong$

$\downarrow$

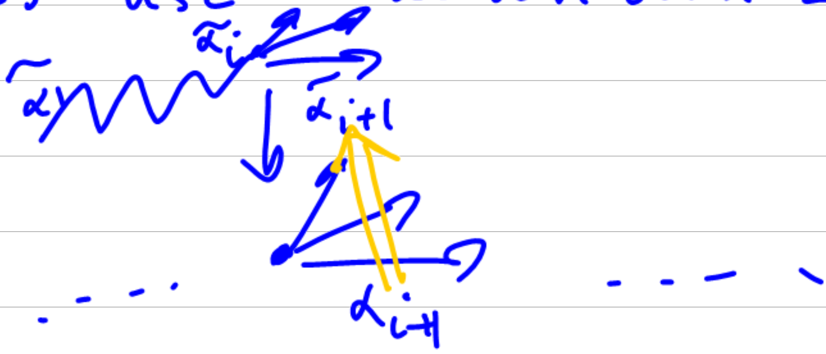
S^1 =

is a cover.

Covers have following path lifting property

Prop^{n} 1: Let $p:\tilde{\Gamma}\to\Gamma$ be a cover of graphs. Given $v\in\tilde{\Gamma}_1$ & path $\alpha_1\alpha_2\ldots\alpha_n$ in Γ with $s(\alpha_1)=p_0(v)$, there exists a unique path $\tilde{\alpha}_1 \tilde{\alpha}_2 \cdots \tilde{\alpha}_n$ in $\tilde{\Gamma}$ with $s(\tilde{\alpha}_1) = v$ lifting $\alpha_1 \ldots \alpha_n$ i.e. $p_1(\tilde{\alpha}_i) = \alpha_i$.

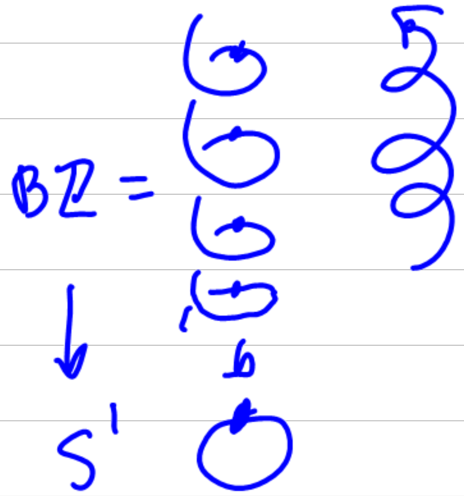
Proof: easy from any picture such as e.g. 1.
Just use induction on n & local bij.



Deck Transformations

Def$^{\underline{n}}$ 2: Let $p: \tilde{\Gamma} \to \Gamma$ be a cover. A deck or covering transformation of p is a graph automorphism $\sigma: \tilde{\Gamma} \to \tilde{\Gamma}$ s.t. $p\sigma = p$ (N.B. $(p\sigma)_0 = p_0 \sigma_0$ & $(p\sigma)_1 = p_1 \sigma_1$).

Eg. 1 again



Winding up $n \in \mathbb{Z}$ "decks" is a deck transformation

$$\mathrm{Gal}(B\mathbb{Z}/S^1) = \mathbb{Z}.$$

Prop$^{\underline{n}}$ 2: Let $p: \tilde{\Gamma} \to \Gamma$ be a cover.

ⓐ If $\sigma: \tilde{\Gamma} \to \tilde{\Gamma}$ is a deck transformation

& $v \in \Gamma$, then σ permutes the fibre $p_0^{-1}(v)$

(b) The set of all deck transf$^{\underline{n}}$s denoted $\mathrm{Gal}(\tilde{\Gamma}/\Gamma)$ is a group under composition called the Galois group of $\hat{\Gamma}/\Gamma$.

Proof: easy ex. $\square$

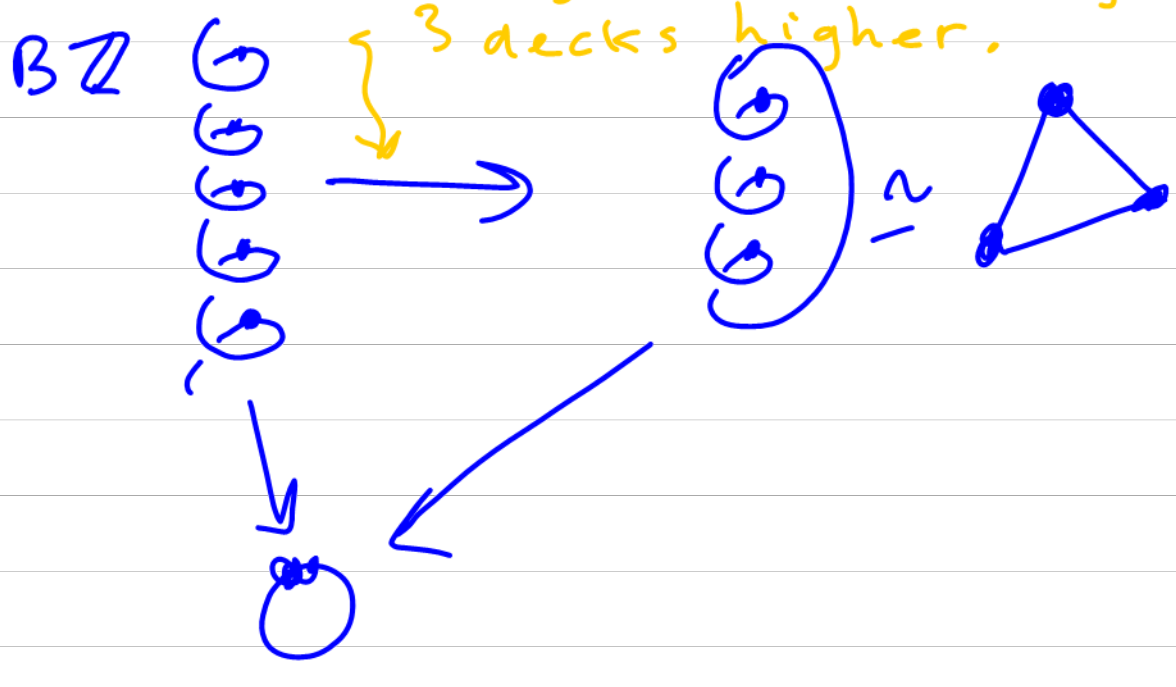
Intermediate Covers

Prop$^{\underline{n}}$-Def$^{\underline{n}}$: Let $p : \tilde{\Gamma} \to \Gamma$ be a cover. An intermediate cover is a factorisation of $p : \tilde{\Gamma} \xrightarrow{q} \Delta \xrightarrow{q'} \Gamma$ where q & hence q' is a cover. In this case $\mathrm{Gal}(\tilde{\Gamma}/\Delta) \leq \mathrm{Gal}(\hat{\Gamma}/\Gamma)$.

Proof: Easy ex.

Eg. 1 again

This map identifies every vertex/edge with vertex/edge 3 decks higher.



Galois Correspondence

Suppose a group G acts on a graph $\tilde{\Gamma}$ i.e. there's a group hom $\varphi: G \to \mathrm{Aut}\,\tilde{\Gamma}$ = group of graph autom. of $\tilde{\Gamma}$.

Prop^n-Def^n 2: We say G acts freely on $\tilde{\Gamma}$ if for any $\sigma \in G - 1$, $v \in \tilde{\Gamma}_0$, $e \in \tilde{\Gamma}_1$ we have $\sigma(v) \neq v$, $\sigma(e) \neq e$ or $\bar{e}$.

In this case, we have a quotient graph $G\backslash\tilde{\Gamma} = (\underbrace{G\backslash\tilde{\Gamma}_0}_{\text{set of } G\text{-orbits of vertices}}, \underbrace{G\backslash\tilde{\Gamma}_1}_{\text{set of } G\text{-orbits of edges}})$ & cover of graphs

$$p: \tilde{\Gamma} \to G\backslash\tilde{\Gamma}$$
$$v \mapsto G\cdot v$$
$$e \mapsto G\cdot e \quad .$$

Proof: ex.

An isom. of intermediate covers of $p: \tilde{\Gamma} \to \Gamma$ is

$$\begin{tikzcd} \tilde{\Gamma} \arrow[r, "q"] \arrow[dr, "q_1"'] & \Delta \arrow[r, "q'"] \arrow[d, "f"] & \Gamma \\ & \Delta_1 \arrow[ur, "q_1'"'] & \end{tikzcd}$$

where f is an isom. of graphs with $fq = q_1$ & $q_1' f = q'$.

Thm (Galois correspondence)

Suppose a group G acts freely on a graph $\tilde{\Gamma}$. Then $G = \mathrm{Gal}(\tilde{\Gamma}/(G\backslash\tilde{\Gamma}))$ & there are inverse bij^{n}s

$$(\tilde{\Gamma} \to \Delta \to G\backslash\tilde{\Gamma}) \longmapsto \mathrm{Gal}(\tilde{\Gamma}/\Delta)$$

$$\left\{\begin{array}{l}\text{Isom. classes of} \\ \text{intermed.} \\ \text{covers } p: \tilde{\Gamma} \to G\backslash\tilde{\Gamma}\end{array}\right\} \rightleftarrows \left\{\begin{array}{c}\text{Subgroups} \\ \text{of } G\end{array}\right\}$$

$$H\backslash\tilde{\Gamma} \longleftarrow\!\shortmid \; H$$

Proof: omitted but see e.g. 1.