Graphs I: Free Groups

Aim Lecture: Intro. group gen. by set X "free" from all rel$^{\underline{n}}$s.

Free Group

Let X = set, $X^{-1} = \{x^{-1} \mid x \in X\}$. Assume $X \cap X^{-1} = \emptyset$. Write $(x^{-1})^{-1} = x$ for $x \in X$.

Def$^{\underline{n}}$ 1: A word in $X^{\pm 1}$ is an expression of form $w = x_1 x_2 \ldots x_n$, $n \geq 0$, $x_i \in X \cup X^{-1}$. Write $w = 1$ if $n = 0$. Let $W(X)$ = set of words in X.

Say $w \in W(X)$ is reduced if it doesn't contain consecutive xx^{-1}, $x \in X \cup X^{-1}$. For $w, w' \in W(X)$, write $w \sim w'$ if w' can be obtained by inserting & deleting 2 letter words of form xx^{-1}, $x \in X \cup X^{-1}$.

e.g. $X = \{x, y, z\}$, $xzx^{-1}x \sim xz \sim xyy^{-1}z$.
$x^{-1}zx$ reduced but not $x^{-1}xz$.

Prop$^{\underline{n}}$-Def$^{\underline{n}}$: The set of equiv. classes $F(X) := W(X)/\sim$ is a group with mult$^{\underline{n}}$ = concatenation of words i.e. $[w_1][w_2] := [w_1 w_2]$.
Call $F(X)$ the free group on X.

Proof: easy ex.

Rem: Usually abuse not$^{\underline{n}}$ & write w for $[w]$.

E.g. 1. $F(\{x\}) = \{\ldots, x^{-1}, 1, x, x^2\} = \langle x \rangle \simeq \mathbb{Z}$.

Basic Properties

Thm (Universal property)

Let $f: X \to G$ be a fn to a group G.

Then $\tilde{f}: F(X) \longrightarrow G$

$$x_1^{\pm 1} x_2^{\pm 1} \ldots x_n^{\pm 1} \longmapsto f(x_1)^{\pm 1} f(x_2)^{\pm 1} \ldots f(x_n)^{\pm 1}$$

is the unique group hom from $F(X) \to G$ extending f.

Proof: easy ex. c.f. lin maps determined on basis.

Normal Form Thm: Every $w \in F(X)$ can be written uniquely as a reduced word.

Proof: NOT OBVIOUS! (Van der Waerden)

Clear w expressible as a reduced word.

Let S = set of reduced words.

Univ. property $\Rightarrow \exists$ group hom $\varphi: F(X) \longrightarrow \mathrm{Perm}\, S$

s.t. for $x \in X$ we have

$$\varphi(x): \underset{\in S}{x_1 \ldots x_n} \longmapsto \begin{cases} x x_1 \ldots x_n, & \text{if } x \neq x_1^{-1} \\ x_2 \ldots x_n, & \text{if } x = x_1^{-1} \end{cases}$$

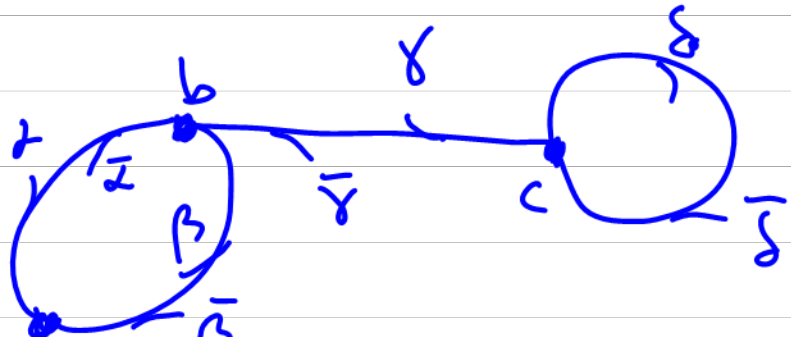
Suffice show reduced words completely determined by its image $[s]$ in $F(X)$.

But $s = \varphi([s])1$

$$x_1 \ldots x_n = \varphi(x_1)\varphi(x_2) \ldots \varphi(x_n) 1$$

$\square$

Graphs

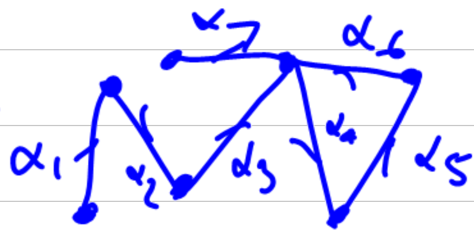


$t(\alpha) = a = s(\bar{\beta})$

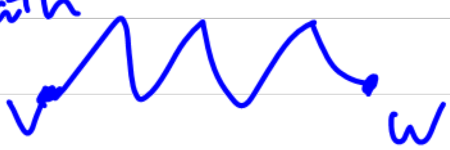
Def$^{\underline{n}}$ 2: A graph $\Gamma = (\Gamma_0, \Gamma_1)$ consists of sets Γ_0 (vertices), Γ_1 (edges) & maps $\bar{\cdot} : \Gamma_1 \to \Gamma_1$, source & target $s, t : \Gamma_1 \to \Gamma_0$ s.t. (a) $\alpha \neq \bar{\alpha}$ but $\bar{\bar{\alpha}} = \alpha$ & (b) $s(\alpha) = t(\bar{\alpha})$ for all $\alpha \in \Gamma_1$.

Hopefully you remember from MATH1081 the def$^{\underline{n}}$s of following:

* path in Γ:



* Γ is connected: For any $v, w \in \Gamma_0$ $\exists$ path



Def$^{\underline{n}}$ 3: A path $\alpha_1 \alpha_2 \dots \alpha_n$ is irreducible if $\alpha_{i+1} \neq \bar{\alpha}_i$ for all i. A graph Γ is a tree if for any $v, w \in \Gamma_0$ there is a unique irred. path $v \rightsquigarrow w$.

Eg. 2. Let $G =$ free group $F(X)$.
Define Cayley graph $BF(X)$ by
$BF(X)_0 = G$, $BF(X)_1 = G \times X \cup G \times X^{-1}$

$$BF(X)_1 \ni e : \quad s(e) = g \xrightarrow{x \in X} gx = t(e)$$

$$\bar{e} : \quad g \xleftarrow{x^{-1}} gx$$

Prop$^{\underline{n}}$: The Cayley graph is a tree.
Proof: This is a restatement of the normal form thm. $\square$

Def$^{\underline{n}}$ 4: Let Δ, Γ be graphs. A morphism of graphs $f : \Delta \to \Gamma$ is pair of set maps $f_0 : \Delta_0 \to \Gamma_0$, $f_1 : \Delta_1 \to \Gamma_1$ s.t.

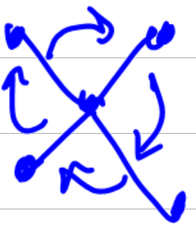
$$f_1 s = s f_1, \quad f_1 t = t f_1, \quad \overline{f_1(e)} = f_1(\bar{e})$$

for $e \in \Delta_1$ & s, t source & target maps in Δ & Γ.

If f_0, f_1 are inclusion maps, we say Δ is a subgraph of Γ. If f_0, f_1 bijective we say f is an automorphism

Eg. 3

Auto.



Subgraph