Cohomology 4: Galois descent

Aim Lecture: See how theory of F-rational forms becomes interesting when we add extra structure to space V e.g. quadratic form or mult$^{\underline{n}}$.

K/F = Galois ext$^{\underline{n}}$, Galois group G.

Rational forms of algebras

Let $\Lambda = M_n(K)$, the K-space of $n\times n$-matrices$/K$. It's a K-algebra too i.e. has K-bilinear mult$^{\underline{n}}$ which makes Λ a ring.

Def$^{\underline{n}}$ 1: An F-rational form of the K-algebra Λ is an F-rational form Λ_0 of the K-space Λ which is closed under mult$^{\underline{n}}$.

E.g. 1. $\Lambda_0 = M_n(F)$ is an F-rational form of Λ.

$\boxed{Q}$ Can you find F-rational forms of the algebra using 1-cocycles?

$\boxed{A}$ YES! But need replace $\mathrm{Aut}_K \Lambda \rightsquigarrow$

$\mathrm{Aut}_{K\text{-alg}} \Lambda :=$ group of K-algebra automorphisms of Λ.

these are K-linear ring isom.

Def$^{\underline{n}}$ 2: Recall $K^* I_n := \left\{ \alpha \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \,\middle|\, \alpha \in K^* \right\}$ is a normal subgroup of $GL_n(K)$. We define the projective linear group to be

$PGL_n(K) := GL_n(K) / K^* I_n$.

K^* G-stable $\Rightarrow$ G acts on $PGL_n(K)$.

<u>Rem</u> (a) $PGL_n(K)$ acts on Λ as follows:
Let $B \in GL_n(K)$ & $\beta \in PGL_n(K)$ be corresp. coset.

$$\beta . \overset{\overset{\Lambda}{\cup\!\!\!\!\!\!\shortmid}}{M} := BMB^{-1}.$$

(b) In this way, $PGL_n(K)$ embeds in $Aut_K \Lambda$. In particular, any 1-cocycle $\gamma_* : G \longrightarrow PGL_n(K)$ gives a 1-cocycle with values in $Aut_K \Lambda$.

(c) A big thm of Skolem-Noether $\Rightarrow$ all K-algebra autom. come from conjugation i.e. $Aut_{K\text{-alg}} \Lambda$ "=" $PGL_n(K)$.

<u>Thm (Descent)</u>: (a) Using the correspondence in Lecture: Cohomology 3, the F-rational forms of the K-algebra $M_n(K)$ are given by the 1-cocycles of G with values in $PGL_n(K)$.

(b) 1-cohomologous cocycles correspond to F-rational forms which are conjugate under $GL_n(K)$.

Proof: Tedious. Just keep track of mult$^{\underline{n}}$ in proofs last lecture. $\square$

Example: Quaternions

$K/F = \mathbb{C}/\mathbb{R}$ so $G = \{1, \sigma = \text{conjug}^{n}\}$.

$\Lambda = M_2(\mathbb{C})$.

Fix preferred action $\sigma.(z_{ij}) = (\overline{z_{ij}})$

Recall 1-cocycle γ_* with values in $PGL_2(\mathbb{C})$ given uniquely by any $\gamma = \gamma_\sigma \in PGL_2(\mathbb{C})$ with $\gamma\overline{\gamma} = 1$ $\therefore$ seek $\Gamma \in GL_2(\mathbb{C})$ s.t.

$\Gamma\overline{\Gamma} = \lambda\begin{pmatrix}1&0\\0&1\end{pmatrix}$, some $\lambda \in \mathbb{C}$.

Check $\Gamma = \begin{pmatrix}0 & i\\ -i & 0\end{pmatrix}$ works.

$$\Gamma\overline{\Gamma} = \begin{pmatrix}0 & i\\ -i & 0\end{pmatrix}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} = \begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}$$

so its coset γ in $PGL_2(\mathbb{C})$ gives 1-cocycle.

Ex. γ is not cohomologous to trivial cocycle.

$\boxed{Q}$ What's corresponding $\mathbb{R}$-rational form Λ_0?

$\boxed{A}$ $\Lambda_0 = \Lambda^G = \Lambda^\sigma$

New G-action corresp. to γ is

$$\sigma.\begin{pmatrix}a & b\\ c & d\end{pmatrix} = \Gamma\overline{\begin{pmatrix}a & b\\ c & d\end{pmatrix}}\Gamma^{-1}$$

$$= \begin{pmatrix}0 & i\\ -i & 0\end{pmatrix}\begin{pmatrix}\bar a & \bar b\\ \bar c & \bar d\end{pmatrix}\begin{pmatrix}0 & i\\ -i & 0\end{pmatrix} = \begin{pmatrix}0 & i\\ -i & 0\end{pmatrix}\begin{pmatrix}-i\bar b & i\bar a\\ -i\bar d & i\bar c\end{pmatrix}$$

$$= \begin{pmatrix}\bar d & -\bar c\\ -\bar b & \bar a\end{pmatrix}$$

$$\therefore \Lambda_0 = \left\{\begin{pmatrix}a & b\\ c & d\end{pmatrix} \,\middle|\, \begin{pmatrix}a & b\\ c & d\end{pmatrix} = \begin{pmatrix}\bar d & -\bar c\\ -\bar b & \bar a\end{pmatrix}\right\}$$

$$= \left\{ \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \middle| \; a, b \in \mathbb{C} \right\}$$

$$= \mathbb{R}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \oplus \mathbb{R}\underbrace{\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}}_{= \underset{\sim}{i}} \oplus \mathbb{R}\underbrace{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}_{= \underset{\sim}{j}} \oplus \mathbb{R}\underbrace{\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}}_{= \underset{\sim}{k}}$$

Λ_0 is Hamilton's algebra of quaternions.

$\Lambda_0 \not\cong M_2(\mathbb{R})$. Indeed,

$$\det \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} = |a|^2 + |b|^2 \geqslant 0 \Rightarrow$$

all non-zero elts of Λ_0 are invertible in $M_2(\mathbb{C})$, hence invertible in Λ_0 ie. Λ_0 is a division ring.

$$\underset{\sim}{i}^2 = \underset{\sim}{j}^2 = \underset{\sim}{k}^2 = -1 \;\&\; \underset{\sim}{i}\,\underset{\sim}{j} = \underset{\sim}{k}$$

Remarks on General Theory of Descent

(a) Theory of central simple algebras $\Rightarrow$ all fin. dim. div rings arise as rational forms of some $M_n(K)$.

(b) The theory here generalises to any extra structure on K-space V given by tensors ie. elts of $V^{\otimes p} \otimes V^{*\otimes q}$.

(c) Another such e.g. is non-degen. symmetric bilinear forms. If K alg. closed, the $V \cong K^n$ with form $(\underset{\sim}{x}, \underset{\sim}{y}) = \sum x_i y_i$.

(d) F-rational forms V_0 have $(\underset{\sim}{x}, \underset{\sim}{y}) \in F$ for all $\underset{\sim}{x}, \underset{\sim}{y} \in V_0$. Classified by $H^1(G, O_n(K))$ where $O_n(K) = \{M \in M_n(K) \mid MM^T = I_n\}$.