Cohomology 3: F-rational forms

Aim Lecture: Some math. objects easier over alg. closed fields e.g. factorising poly. Naturally raises Q studied today: can you obtain results /F from results /$\bar{F}$?

F-rational Forms

Let K/F = Galois ext^{n}, Gal group G.

V = K-space, $\dim_K V < \infty$

Def^{n} 1: A G-equivariant action on the K-space V is a G-action on the additive group V s.t.

$$\sigma.(\alpha v) = \sigma(\alpha)(\sigma.v) \quad \forall \sigma \in G, \alpha \in K, v \in V.$$

E.g 1. $V = K^n$. G-action $\sigma\begin{pmatrix}\alpha_1\\ \vdots \\ \alpha_n\end{pmatrix} := \begin{pmatrix}\sigma(\alpha_1)\\ \vdots \\ \sigma(\alpha_n)\end{pmatrix}$

Check: $\sigma.(\alpha v) = \sigma\begin{pmatrix}\alpha\alpha_1\\ \vdots \\ \alpha\alpha_n\end{pmatrix} \overset{v}{=} \begin{pmatrix}\sigma(\alpha)\sigma(\alpha_1)\\ \vdots \\ \sigma(\alpha)\sigma(\alpha_n)\end{pmatrix} = \sigma(\alpha)\sigma.v$

N.B. σ is F-linear $\because$ $\alpha \in F \Rightarrow \sigma(\alpha) = \alpha$.

Prop^{n}-Def^{n}: (a) An F-rational form for K-space V is a subspace V_0 of F-space V s.t. any of the following equiv. cond^{n}s hold

(i) There's an F-basis $\{v_1, \ldots, v_n\}$ for V_0 which is a K-basis for V.

(ii) Any F-basis for V_0 is a K-basis for V.

(b) In this case, using not^{n} in (i), the following defines a G-equiv. action on V

$\sigma.(\alpha_1 v_1 + \ldots + \alpha_n v_n) = \sigma(\alpha_1)v_1 + \ldots + \sigma(\alpha_n)v_n$

which is independent of choice of F-basis $\{v_i\}$.

(c) The fixed F-subspace

$V^G := \{v \in V \mid \sigma(v) = v, \text{ for all } \sigma \in G\} = V_0$.

Proof: ex □.

E.g. 1 again. $F^n \subseteq K^n$ an F-rational form for K^n. G-equiv. action in (b) is one given in e.g. 1. Clear $(K^n)^G = F^n$ here.

$\boxed{\text{Speiser's Lemma}}$

Following converse result shows F-rational forms correspond to G-equiv. actions.

Lemma (Speiser): Given a G-equiv. action on K-space V, $V_0 := V^G$ is an F-rational form for V & the G-equiv. action is the same as that given in Prop^n-Def^n (b).

Proof: only in case $G = \langle \sigma \rangle$ order n & F contains primitive n-th root ζ of 1 so $K = F(\alpha)$ where $\alpha^n \in F$ & $\sigma(\alpha) = \zeta\alpha$.

Note $\dim_F V \overset{ex}{=} \dim_K V . [K:F] = n \dim_K V$

$V_0 = V^G = V^\sigma = \{v \in V \mid \sigma.v = v\}$

$= 1$-e-space of F-linear map $\sigma : V \to V : v \mapsto \sigma.v$

More gen. define $V_i = \zeta^i$-e-space &

note $\sigma^n = 1 \Rightarrow V \overset{\circledast}{=} V_0 \oplus V_1 \oplus \dots \oplus V_{n-1}$.

For $v \in V_i$, $\sigma.(\alpha^j v) = \sigma(\alpha^j)(\sigma.v) =$

$\zeta^j \alpha^j \zeta^i v = \zeta^{i+j} \alpha^j v \Rightarrow \alpha^j v \in V_{i+j}$.

Hence mult$^{\underline{n}}$ by α cyclically permutes e-spaces & $\circledast \Rightarrow$

$$V = V_0 \oplus \alpha V_0 \oplus \dots \oplus \alpha^{n-1} V_0 .$$

Now $K = F(\alpha)$ so any F-basis for V_0 is a K-basis for V.

ex. show last statement of lemma $\square$

Relation to 1-cocycles

$\boxed{Q}$ How do two G-equiv. actions differ?

$\boxed{A}$ Fix "preferred" G-equiv. action

$\varphi : G \to \operatorname{Aut} V$,

Let $\operatorname{Aut}_K V =$ group of K-linear K-space isom. $V \to V$

G acts on $\operatorname{Aut}_K V$ by $\sigma.\gamma := \varphi(\sigma) \circ \gamma \circ \varphi(\sigma^{-1})$

Note $\sigma.\gamma \in \operatorname{Aut}_K V$:

$$(\sigma.\gamma)(\alpha v) = \varphi(\sigma) \circ \gamma \circ \varphi(\sigma^{-1})(\alpha v)$$

$$\underset{K}{\alpha} \quad \underset{V}{v} \quad = [\varphi(\sigma) \circ \gamma](\sigma^{-1}(\alpha)\, \sigma^{-1}.v)$$

$$= [\varphi(\sigma)](\sigma^{-1}(\alpha)\, \gamma(\sigma^{-1}.v))$$

$$= \alpha\, (\sigma.\, \gamma(\sigma^{-1}.v))$$

$$= \alpha\, (\sigma.\gamma)(v)$$

Consider another G-equiv. action
$\psi: G \to \mathrm{Aut}\, V$. Define
$\gamma_\sigma = \psi(\sigma)\varphi(\sigma^{-1})$.

<u>Prop$^{\underline{n}}$ 1</u>: ⓐ We obtain a 1-cocycle
$\gamma_*: G \to \mathrm{Aut}_k V$.
ⓑ Conversely, given any 1-cocycle
$\delta_*: G \to \mathrm{Aut}_k V$, $\psi: G \to \mathrm{Aut}\, V: \sigma \mapsto \delta_\sigma \circ \varphi(\sigma)$ defines a G-equiv. action.

Proof: ⓐ only . ex check $\gamma_\sigma \in \mathrm{Aut}_k V$.

$$\gamma_{\sigma\tau} = \psi(\sigma\tau)\varphi(\tau^{-1}\sigma^{-1}) = \psi(\sigma)\psi(\tau)\varphi(\tau^{-1})\varphi(\sigma^{-1})$$

$$= \underbrace{\psi(\sigma)\varphi(\sigma^{-1})}_{\gamma_\sigma}\varphi(\sigma)\underbrace{\psi(\tau)\varphi(\tau^{-1})}_{\gamma_\tau}\varphi(\sigma^{-1})$$

$$= \gamma_\sigma(\sigma . \gamma_\tau). \qquad \square$$

<u>Scholium</u>: Suppose 1-cocycle $\delta_* \longleftrightarrow$ G-equiv. action $\psi \longleftrightarrow$ F-rational form $V_0 = V^G$.
For $a \in \mathrm{Aut}_k V$, 1-cohomologous cocycle
$(a.\delta_*)_\sigma = a\,\delta_\sigma(\sigma . a^{-1}) \longleftrightarrow$ G-equiv. action
$\sigma \mapsto a\,\delta_\sigma\varphi(\sigma)\,a^{-1}\varphi(\sigma^{-1})\varphi(\sigma) = a\,\delta_\sigma\varphi(\sigma)a^{-1}$
$\therefore$ corresp. F-rational form is aV_0.

<u>Rem</u>: Nothing interesting here $\because$ $H^1(G, GL_n(k)) = 1$.