THE UNIVERSITY OF NEW SOUTH WALES

SCHOOL OF MATHEMATICS AND STATISTICS

Semester 2 2014

MATH1231
MATHEMATICS 1B

(1) TIME ALLOWED – TWO (2) HOURS

(2) TOTAL NUMBER OF QUESTIONS – 4

(3) ANSWER ALL QUESTIONS

(4) THE QUESTIONS ARE OF EQUAL VALUE

(5) ANSWER **EACH** QUESTION IN A **SEPARATE** BOOK

(6) THIS PAPER MAY BE RETAINED BY THE CANDIDATE

(7) **ONLY** CALCULATORS WITH AN AFFIXED "UNSW APPROVED" STICKER MAY BE USED

(8) A SHORT TABLE OF INTEGRALS and A STANDARD NORMAL TABLE ARE APPENDED ON THE LAST PAGES

All answers must be written in ink. Except where they are expressly required pencils may only be used for drawing, sketching or graphical work.

Use a SEPARATE book clearly marked Question 1

1. i) By expanding $\sin(A+B)+\sin(A-B)$, or otherwise, find $I_1 = \displaystyle\int \sin(5x)\cos(x)\,dx$.

ii) Evaluate the integral $I_2 = \displaystyle\int_0^1 \frac{x^2}{\sqrt{4-x^2}}\,dx$.

iii) Use appropriate tests to determine whether each of the following series converges or diverges

a) $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n(\ln\ n)^3}$,

b) $\displaystyle\sum_{n=1}^{\infty} \frac{2}{2^n+3^n}$.

iv) Let $S = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \ :\ z^2 = x^2 + y^2 \right\}$.

a) Prove that S is closed under scalar multiplication.

b) Prove that S is **not** a subspace of $\mathbb{R}^3$.

v) Let $A = \begin{pmatrix} 0 & -1 \\ 2 & 3 \end{pmatrix}$.

a) Find the eigenvalues and eigenvectors for the matrix A.

b) Write down an invertible matrix M and a diagonal matrix D such that

$$D = M^{-1}AM.$$

vi) Let

$$A = \begin{pmatrix} 1 & 2 & 2 & -1 \\ 2 & 4 & 10 & -44 \\ 3 & -3 & -3 & 24 \\ 1 & 2 & 1 & 6 \end{pmatrix}$$

Using the MAPLE output below, find a basis for $\ker(A)$.

```
> with(LinearAlgebra):
> A := <<1,2,3,1>|<2,4,-3,2>|<2,10,-3,1>|<-1,-44,24,6>>;
```

$$\begin{bmatrix} 1 & 2 & 2 & -1 \\ 2 & 4 & 10 & -44 \\ 3 & -3 & -3 & 24 \\ 1 & 2 & 1 & 6 \end{bmatrix}$$

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> ReducedRowEchelonForm(A);
```

$$\begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -7 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Use a SEPARATE book clearly marked Question 2

2. i) Consider the initial value problem $\dfrac{dy}{dx} + (2 + \dfrac{1}{x})y = \dfrac{2}{x}$, with $y(1) = 0$, defined for $x > 0$.

a) Show that an integrating factor for this equation is xe^{2x}.

b) Hence solve the initial value problem.

ii) Find the general solution to $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + 2y = 20e^{2x}$.

iii) Consider the MAPLE session:

```
> a:=n->n^n/n!*(x-1)^n;
```

$$a := n \to \frac{n^n (x-1)^n}{n!}$$

```
> a(n+1);
```

$$\frac{(n+1)^{(n+1)}(x-1)^{(n+1)}}{(n+1)!}$$

```
> limit(a(n+1)/a(n),n=infinity);
```

$$ex - e$$

Using MAPLE session above, or otherwise, find the open interval of convergence $I = (a, b)$ for the power series

$$\sum_{n=1}^{\infty} \frac{n^n (x-1)^n}{n!}.$$

iv) Consider the set S consisting of the vectors $\mathbf{v_1} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$, $\mathbf{v_2} = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$ from $\mathbb{R}^3$ and let $\mathbf{u} = \begin{pmatrix} 1 \\ -1 \\ 12 \end{pmatrix}$.

a) Find scalars λ and μ such $\mathbf{u} = \lambda \mathbf{v_1} + \mu \mathbf{v_2}$.

b) A linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ has $\mathbf{v_1}, \mathbf{v_2}$ as eigenvectors with eigenvalues 2 and -1, respectively.

α) Find $T(\mathbf{u})$ as a linear combination of $\mathbf{v_1}, \mathbf{v_2}$.

β) Denote $T(T(\mathbf{u}))$ by $T^2(\mathbf{u})$, $T(T(T(\mathbf{u})))$ by $T^3(\mathbf{u})$, and so on. Express $T^n(\mathbf{u})$ as a linear combination of $\mathbf{v_1}, \mathbf{v_2}$, where n is a positive integer.

Please see over ...

v) The two most popular soft drinks in Old South Wales are AppleAde and BananAde. Assume that no-one in Old South Wales likes both of these drinks equally (that is, everyone has a preference for one or the other). Past statistics show that 50 % of the population prefer AppleAde.

Last month the manufacturer advertised AppleAde on television for a week. After that, a survey was conducted by taking a random sample of 100 people. Of the 100 people sampled, 60 preferred AppleAde and 40 preferred BananAde.

a) Assuming that the advertising had **no effect** on people's preferences, write down an expression for the tail probability that 60 or more people preferred AppleAde in a sample of 100.

b) Use the normal approximation to the binomial to calculate the tail probability in (a), giving your answer to 3 decimal places.

c) Giving reasons, is there evidence that the advertising campaign increased the percentage of the population that prefer AppleAde?

Use a SEPARATE book clearly marked Question 3

3. i) Let $\mathbb{P}_1$ denote the space of all polynomials of degree less or equal to 1. Prove that function $T : \mathbb{R}^3 \to \mathbb{P}_1$ defined by

$$T(\mathbf{v}) = 2a + (b - c)x, \quad \text{for all } \mathbf{v} = \begin{pmatrix} a \\ b \\ c \end{pmatrix},$$

is a linear transformation.

ii) Let S be the set of vectors $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}, \mathbf{v_4}, \mathbf{v_5}\}$ given by

$$S = \left\{ \begin{pmatrix} 21 \\ 12 \\ -6 \\ -31 \\ -12 \end{pmatrix}, \begin{pmatrix} 11 \\ 20 \\ -14 \\ -29 \\ -21 \end{pmatrix}, \begin{pmatrix} 18 \\ -72 \\ 60 \\ 50 \\ 78 \end{pmatrix}, \begin{pmatrix} 29 \\ -52 \\ 46 \\ 21 \\ 57 \end{pmatrix}, \begin{pmatrix} 16 \\ -17 \\ -31 \\ 28 \\ -21 \end{pmatrix} \right\}.$$

Using the following MAPLE output, answer the questions below. Give reasons.

```
> with(LinearAlgebra):
> M := <<21, 12, -6, -31, -12>|<11, 20, -14, -29, -21>
           |<18, -72, 60, 50, 78>|<29, -52, 46, 21, 57>
           |<16, -17, -31, 28, -21>>;
```

$$\begin{bmatrix} 21 & 11 & 18 & 29 & 16 \\ 12 & 20 & -72 & -52 & -17 \\ -6 & -14 & 60 & 46 & -31 \\ -31 & -29 & 50 & 21 & 28 \\ -12 & -21 & 78 & 57 & -21 \end{bmatrix}$$

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> GaussianElimination(M);
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$$\begin{bmatrix} 21 & 11 & 18 & 29 & 16 \\ 0 & \frac{96}{7} & -\frac{576}{7} & -\frac{480}{7} & -\frac{183}{7} \\ 0 & 0 & 0 & 0 & -\frac{377}{8} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

a) Find a basis for $\text{span}(S)$.
b) Write down the dimension of $\text{span}(S)$.

iii) Employment data at a large company reveal that 40 % of the employees are thirty years of age or younger; 60 % are older than thirty. All employees belong to exactly one of the types: full-time, part-time or casual. Among those who are thirty years of age or younger, 25 % are full-time; 15 % are part-time; the others are casual. Among those who are older than thirty, 45 % are full-time; 20 % are part-time; the others are casual.

a) What proportion of the employees are full-time?
b) What is the probability that a randomly chosen employee is older than thirty given that the employment type of the employee is full-time?

iv) The discrete random variable X can only take the values $-2, -1, 1, 2, 5$. The probability distribution for X is given in the following table:

x	-2	-1	1	2	5
$P(X = x)$	0.3	$4c$	0.35	$6c$	0.05

Another random variable Y is defined by $Y = X^2$.

a) Find the value of c.
b) Calculate $P(Y = 4)$.

v) The density function f for a random variable X is defined by

$$f(x) = \begin{cases} 3x^2 & \text{for} \quad 0 < x \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

a) Find the cumulative probability density function F corresponding to f.
b) Calculate $E(X)$ for the probability density function f.
c) Calculate $\text{Var}(X)$.

vi) Let $B = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ be a set of three non-zero vectors in $\mathbb{R}^3$.

a) State the definition for the set B to be a linearly independent set.
b) Prove that if B is an orthogonal set then B is linearly independent.
c) Hence explain why any orthogonal set of 3 non-zero vectors in $\mathbb{R}^3$ forms a basis for $\mathbb{R}^3$.

Use a SEPARATE book clearly marked Question 4

4. i) A right circular cone with base radius r cm and a perpendicular height h cm is formed by rotating the line segment given by

$$y = \frac{r}{h}\,x, \qquad 0 \le x \le h,$$

about the x-axis.

a) Use calculus to show that the surface area S of the curved surface of the cone is given by

$$S = \pi r\sqrt{r^2 + h^2}.$$

b) Find the partial derivatives $\dfrac{\partial S}{\partial r}$ and $\dfrac{\partial S}{\partial h}$.

c) If the values of r and h are measured to be 3.0 and 4.0 cms respectively and each of these measurements is made with an error whose absolute value is at most 0.05 cm, then use the total differential approximation of S to estimate the maximum absolute error in the measured value of the total surface area S.

ii) During the winter the daytime temperature in the Physics Theatre is maintained at 20°C. The heating is turned off at 10pm and turned on again at 6am. On a certain day, the temperature inside the Theatre at 11pm was found to be 18°C. The outside temperature was found to be constant throughout the night at 10°C. Let $P(t)$ be the temperature (in °C) in the Physics Theatre at time t (in hours), from 10pm. The rate of cooling of the air inside the Theatre can be modelled by the differential equation

$$\frac{dP}{dt} = -k(P - 10),$$

where k is a positive constant. (Do not prove this.)

a) By first solving the differential equation, show that $k = \ln(5/4)$.

b) Find, to two decimal places, the temperature inside the Physics Theatre when the heating was turned on at 6am.

iii) The n^{th} term of the sequence $\{a_n\}$ is given by

$$a_n := \begin{cases} n^{1/n}/2, & \text{for } n \text{ odd,} \\ 1/2, & \text{for } n \text{ even.} \end{cases}$$

a) Explain briefly why

$$\frac{1}{2} \leq a_n \leq \frac{n^{1/n}}{2}, \text{ for } n \geq 1.$$

b) Prove that

$$\lim_{n\to\infty} n^{1/n} = 1.$$

c) Hence find

$$\lim_{n\to\infty} a_n.$$

Give reasons for your answer.

d) Does the series

$$\sum_{n=2}^{\infty} (-1)^n a_n$$

converge or diverge? Name any test that you use.

iv) Let $f(x)$ be a function satisfying

$$f(0) = 0, \quad f'(0) = 1, \quad f''(0) = -2 \text{ and } |f'''(x)| \leq 6, \text{ for } 0 \leq x \leq 1.$$

a) Write down the second Taylor polynomial $p_2(x)$ of $f(x)$ about $x = 0$.

b) The function $f(x)$ is approximated by $p_2(x)$, for x in the interval $0 \leq x \leq 1$.
Using Taylor's Theorem with Lagrange remainder formula, show that

$$\frac{1}{8} \leq f(1/2) \leq \frac{3}{8}.$$

c) Using (a), or otherwise, find

$$\lim_{x\to 0^+} \frac{x - f(x)}{x^2}.$$

BASIC INTEGRALS

$$\int \frac{1}{x}\,dx = \ln|x| + C = \ln|kx|, \qquad C = \ln k$$
$$\int e^{ax}\,dx = \frac{1}{a}e^{ax} + C$$
$$\int a^x\,dx = \frac{1}{\ln a}a^x + C, \qquad a \neq 1$$
$$\int \sin ax\,dx = -\frac{1}{a}\cos ax + C$$
$$\int \cos ax\,dx = \frac{1}{a}\sin ax + C$$
$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax + C$$
$$\int \operatorname{cosec}^2 ax\,dx = -\frac{1}{a}\cot ax + C$$
$$\int \tan ax\,dx = \frac{1}{a}\ln|\sec ax| + C$$
$$\int \cot ax\,dx = \frac{1}{a}\ln|\sin ax| + C$$
$$\int \sec ax\,dx = \frac{1}{a}\ln|\sec ax + \tan ax| + C$$
$$\int \sinh ax\,dx = \frac{1}{a}\cosh ax + C$$
$$\int \cosh ax\,dx = \frac{1}{a}\sinh ax + C$$
$$\int \operatorname{sech}^2 ax\,dx = \frac{1}{a}\tanh ax + C$$
$$\int \operatorname{cosech}^2 ax\,dx = -\frac{1}{a}\coth ax + C$$
$$\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$$
$$\begin{aligned}\int \frac{dx}{a^2-x^2} &= \frac{1}{a}\tanh^{-1}\frac{x}{a} + C, \qquad |x| < a \\ &= \frac{1}{a}\coth^{-1}\frac{x}{a} + C, \qquad |x| > a > 0 \\ &= \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + C, \qquad x^2 \neq a^2\end{aligned}$$
$$\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a} + C$$
$$\int \frac{dx}{\sqrt{x^2+a^2}} = \sinh^{-1}\frac{x}{a} + C$$
$$\int \frac{dx}{\sqrt{x^2-a^2}} = \cosh^{-1}\frac{x}{a} + C, \qquad x \geqslant a > 0$$

Standard normal probabilities $P(Z \leq z)$

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
−2.9	0.0019	0.0018	0.0018	0.0017	0.0016	0.0016	0.0015	0.0015	0.0014	0.0014
−2.8	0.0026	0.0025	0.0024	0.0023	0.0023	0.0022	0.0021	0.0021	0.0020	0.0019
−2.7	0.0035	0.0034	0.0033	0.0032	0.0031	0.0030	0.0029	0.0028	0.0027	0.0026
−2.6	0.0047	0.0045	0.0044	0.0043	0.0041	0.0040	0.0039	0.0038	0.0037	0.0036
−2.5	0.0062	0.0060	0.0059	0.0057	0.0055	0.0054	0.0052	0.0051	0.0049	0.0048
−2.4	0.0082	0.0080	0.0078	0.0075	0.0073	0.0071	0.0069	0.0068	0.0066	0.0064
−2.3	0.0107	0.0104	0.0102	0.0099	0.0096	0.0094	0.0091	0.0089	0.0087	0.0084
−2.2	0.0139	0.0136	0.0132	0.0129	0.0125	0.0122	0.0119	0.0116	0.0113	0.0110
−2.1	0.0179	0.0174	0.0170	0.0166	0.0162	0.0158	0.0154	0.0150	0.0146	0.0143
−2.0	0.0228	0.0222	0.0217	0.0212	0.0207	0.0202	0.0197	0.0192	0.0188	0.0183
−1.9	0.0287	0.0281	0.0274	0.0268	0.0262	0.0256	0.0250	0.0244	0.0239	0.0233
−1.8	0.0359	0.0351	0.0344	0.0336	0.0329	0.0322	0.0314	0.0307	0.0301	0.0294
−1.7	0.0446	0.0436	0.0427	0.0418	0.0409	0.0401	0.0392	0.0384	0.0375	0.0367
−1.6	0.0548	0.0537	0.0526	0.0516	0.0505	0.0495	0.0485	0.0475	0.0465	0.0455
−1.5	0.0668	0.0655	0.0643	0.0630	0.0618	0.0606	0.0594	0.0582	0.0571	0.0559
−1.4	0.0808	0.0793	0.0778	0.0764	0.0749	0.0735	0.0721	0.0708	0.0694	0.0681
−1.3	0.0968	0.0951	0.0934	0.0918	0.0901	0.0885	0.0869	0.0853	0.0838	0.0823
−1.2	0.1151	0.1131	0.1112	0.1093	0.1075	0.1056	0.1038	0.1020	0.1003	0.0985
−1.1	0.1357	0.1335	0.1314	0.1292	0.1271	0.1251	0.1230	0.1210	0.1190	0.1170
−1.0	0.1587	0.1562	0.1539	0.1515	0.1492	0.1469	0.1446	0.1423	0.1401	0.1379
−0.9	0.1841	0.1814	0.1788	0.1762	0.1736	0.1711	0.1685	0.1660	0.1635	0.1611
−0.8	0.2119	0.2090	0.2061	0.2033	0.2005	0.1977	0.1949	0.1922	0.1894	0.1867
−0.7	0.2420	0.2389	0.2358	0.2327	0.2296	0.2266	0.2236	0.2206	0.2177	0.2148
−0.6	0.2743	0.2709	0.2676	0.2643	0.2611	0.2578	0.2546	0.2514	0.2483	0.2451
−0.5	0.3085	0.3050	0.3015	0.2981	0.2946	0.2912	0.2877	0.2843	0.2810	0.2776
−0.4	0.3446	0.3409	0.3372	0.3336	0.3300	0.3264	0.3228	0.3192	0.3156	0.3121
−0.3	0.3821	0.3783	0.3745	0.3707	0.3669	0.3632	0.3594	0.3557	0.3520	0.3483
−0.2	0.4207	0.4168	0.4129	0.4090	0.4052	0.4013	0.3974	0.3936	0.3897	0.3859
−0.1	0.4602	0.4562	0.4522	0.4483	0.4443	0.4404	0.4364	0.4325	0.4286	0.4247
−0.0	0.5000	0.4960	0.4920	0.4880	0.4840	0.4801	0.4761	0.4721	0.4681	0.4641
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986