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Chapter 8: Eigenvalues and eigenvectors

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Motivating example: reflections

Example

Show that the linear map $T_A : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ associated to $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is reflection about some line.

Soln

To understand the map T, useful to change the co-ord axes to $y = x$ and $y = -x$. Then T is reflection about a co-ord axis.

Q Given A, what are good co-ordinates to understand T_A?

A The theory of eigenvalues and eigenvectors gives an answer.

Definition of Eigenvectors and Eigenvalues

Definition (For Linear Maps)

Let V = vector space $/\mathbb{F}$ & $T : V \to V$ be linear. Then if a scalar $\lambda \in \mathbb{F}$ and a non-zero vector $\mathbf{v} \in V$ satisfy

$$T(\mathbf{v}) = \lambda \mathbf{v},$$

then $\lambda \in \mathbb{F}$ is called an **eigenvalue** of T and $\mathbf{v}$ is called an **eigenvector** of T for the eigenvalue λ.

Note: The domain and codomain are the same vector space.

Definition (For Matrices)

Let A be an $n \times n$ square matrix. Then if a scalar $\lambda \in \mathbb{F}$ and non-zero vector $\mathbf{x} \in \mathbb{F}^n$ satisfy

$$A\mathbf{x} = \lambda \mathbf{x},$$

then λ is called an **eigenvalue** of A and $\mathbf{x}$ is called an **eigenvector** of A for the eigenvalue λ.

Eigenvalues for reflection

Example

Some e-vectors & e-values for the reflection matrix $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

$A\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 1\begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \therefore \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an e-vector with e-value 1.

$A\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} = -\begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \therefore$ is an e-vector with e-value -1.

Remark

1. Any non-zero vector on line of reflection $y = x$ is an e-vector with e-value 1.
2. Sim, any non-zero vector on the orthogonal line $y = -x$ is an e-vector with e-value -1.
3. No other e-vectors so $1, -1$ are the only e-values for A.
4. **Upshot** Eigenvectors give the good co-ord axes!

Eigenspaces

Q How do you find e-vectors & e-values?

Partial A Given $\lambda \in \mathbb{F}$, find e-vectors with e-value λ using:

Theorem-Defn

The λ-**eigenspace** of a square matrix A is the subspace $\ker(A - \lambda I)$. The e-vectors of A with e-value λ are precisely the non-zero vectors of $\ker(A - \lambda I)$.

Proof.

$\mathbf{v} \neq \mathbf{0}$ is an e-vector with e-value $\lambda \iff$

□

Corollary

λ is an e-value for A iff $\ker(A - \lambda I) \neq \mathbf{0}$.

Examples & Remarks on e-vectors

E.g. The 1-eigenspace of the reflection matrix $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is

$\ker(A - 1I) =$

Rem

- The 0-e-space is just $\ker A$ so the e-vectors with e-value 0 are the non-zero vectors in $\ker A$.
- $\ker(A - \lambda I)$ is closed under scalar multn i.e. $\mathbf{v}$ an e-vector with e-value $\lambda \implies$ so is any non-zero scalar multiple of $\mathbf{v}$.

Finding e-values

To find e-values, we use

Theorem

λ is an e-value of a square matrix A iff $\det(A - \lambda I) = 0$*.*

Proof.

λ is an e-value for $A \iff \ker(A - \lambda I) \neq \mathbf{0} \iff$ solns to $(A - \lambda I)\mathbf{x} = \mathbf{0}$ are not unique $\iff (A - \lambda I)$ is not invertible $\iff \det(A - \lambda I) = 0$.

2×2 Matrices

Example

Find the eigenvalues and eigenvectors of the matrix $\begin{pmatrix} 3 & -2 \\ -1 & 2 \end{pmatrix}$.

Solution

Solution (Continued)

Solution (Continued)

This example, e-values are distinct. Have a basis of e-vectors $\begin{pmatrix} 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ which gives good co-ords to study A.

Characteristic polynomial

Theorem

- *Let $A = n \times n$ matrix $/\mathbb{C}$. Then $\det(A - \lambda I)$ is a polynomial of degree n in λ. The polynomial $\det(A - \lambda I)$ is called the* **characteristic polynomial** *for A.*
- *An $n \times n$ matrix always has n e-values in $\mathbb{C}$ (not necessarily distinct).*
- *If all n e-values are distinct, and $B = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ are e-vectors with distinct e-values, then B is a basis for $\mathbb{C}^n$ (and for $\mathbb{R}^n$ if the vectors are real).*

Warning If e-values are not distinct then there may not be a basis of e-vectors. This happens for $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

Example

Find the eigenvalues and eigenvectors of the matrix $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$.

Solution

Hence, the eigenvalue is 2.

Solution (Continued)

Compute 2-e-space $\ker(A-2I)$*. Solve*

$$(A-2I)\mathbf{v}=\mathbf{0} \iff \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\mathbf{v}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

$\mathbf{v}$ *can be anything i.e. 2-e-space is* $\mathbb{R}^2$*.*
Hence any non-zero vector in $\mathbb{R}^2$ *is an e-vector. The e-vectors are*

$$s\begin{pmatrix} 1 \\ 0 \end{pmatrix}+t\begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

not both s *and* t *are* 0*.*

Here the e-values are equal. Still have a basis of e-vectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Example

Find the eigenvalues and eigenvectors of the matrix $\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$.

Solution

Let $A = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$.
E-values of A are the solns to characteristic eqn

$$\begin{aligned} \det(A - \lambda I) &= 0 \\ \det \begin{pmatrix} 1-\lambda & 1 \\ -1 & 1-\lambda \end{pmatrix} &= 0 \\ (1-\lambda)(1-\lambda) - (-1) &= 0 \\ (\lambda - 1)^2 &= -1 \\ \lambda = 1 \pm i \end{aligned}$$

E-values are $1 + i$ *and* $1 - i$.

Solution (Continued)

The e-vectors $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ *of* A *for* $\lambda = 1 + i$ *are non-zero solns to*

$$(A - (1+i)I)\mathbf{v} = \mathbf{0}$$

which is $\begin{pmatrix} 1-(1+i) & 1 \\ -1 & 1-(1+i) \end{pmatrix} \mathbf{v} = \begin{pmatrix} -i & 1 \\ -1 & -i \end{pmatrix} \mathbf{v} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$

Reduce the augmented matrix with the right hand zero column omitted:

$$\begin{pmatrix} -i & 1 \\ -1 & -i \end{pmatrix} \xrightarrow{R_2 = R_2 + iR_1} \begin{pmatrix} -i & 1 \\ 0 & 0 \end{pmatrix}$$

Hence $-iv_1 + v_2 = 0$*. Put* $v_2 = t$ *so* $v_1 = -it$*. E-vectors are*
$t \begin{pmatrix} -i \\ 1 \end{pmatrix}, \quad t \neq 0.$

Solution (Continued)

E-vectors $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ *of A for* $\lambda = 1 - i$ *are non-zero solns to*

$$(A - (1-i)I)\mathbf{v} = \mathbf{0}$$

which is $\begin{pmatrix} 1-(1-i) & 1 \\ -1 & 1-(1-i) \end{pmatrix} \mathbf{v} = \begin{pmatrix} i & 1 \\ -1 & i \end{pmatrix} \mathbf{v} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$

Reduce the augmented matrix with the right hand zero column omitted:

$$\begin{pmatrix} i & 1 \\ -1 & i \end{pmatrix} \xrightarrow{R_2 = R_2 - iR_1} \begin{pmatrix} i & 1 \\ 0 & 0 \end{pmatrix}$$

Hence $iv_1 + v_2 = 0$. *E-vectors are* $t \begin{pmatrix} i \\ 1 \end{pmatrix}, \quad t \neq 0.$

Here, e-values are distinct & not real. Again have basis of e-vectors
$\begin{pmatrix} -i \\ 1 \end{pmatrix}, \begin{pmatrix} i \\ 1 \end{pmatrix}.$

Higher Order Square Matrices

Example

Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{pmatrix} -3 & 4 & 2 & -3 \\ 2 & 12 & -4 & 2 \\ -5 & 12 & 2 & -1 \\ -15 & 4 & 2 & 9 \end{pmatrix}.$$

Solution (Use Maple)

```
> with(LinearAlgebra):
> A := <<-3, 2, -5, -15>|<4, 12, 12, 4>|<2, -4, 2, 2>|<-3,
2, -1, 9>>;
```

$$\begin{bmatrix} -3 & 4 & 2 & -3 \\ 2 & 12 & -4 & 2 \\ -5 & 12 & 2 & -1 \\ -15 & 4 & 2 & 9 \end{bmatrix}$$

Solution (Continued)

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> Eigenvectors(A);
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$$\begin{bmatrix} 4 \\ -4 \\ 12 \\ 8 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & \frac{1}{2} & 2 \\ 3 & 1 & \frac{1}{2} & 3 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

Check these directly!

Diagonal matrices are easy to multiply

Multiplication of diagonal matrices is easy.

$$C = \begin{pmatrix} c_1 & 0 & \dots & 0 \\ 0 & c_2 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & c_n \end{pmatrix}, \quad D = \begin{pmatrix} d_1 & 0 & \dots & 0 \\ 0 & d_2 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & d_n \end{pmatrix}$$

Proposition

i) $CD = \begin{pmatrix} c_1 d_1 & 0 & \dots & 0 \\ 0 & c_2 d_2 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & c_n d_n \end{pmatrix}$, ii) $D^k = \begin{pmatrix} d_1^k & 0 & \dots & 0 \\ 0 & d_2^k & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & d_n^k \end{pmatrix}$

Note that i) $\implies$ ii) by induction.

Examples of multiplying diagonal matrices

Propn clear from any **eg**

Diagonalisation: motivation

Key Fact A square matrix A can be recovered from a basis of e-vectors & corresponding e-values.

Example

Let $A = 3 \times 3$ matrix with e-values 1, 3, -2 & the corresponding basis of e-vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$. Find A.

Let $M = (\mathbf{v}_1|\mathbf{v}_2|\mathbf{v}_3)$ and $D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -2 \end{pmatrix}$.

$$\begin{aligned} AM &= A(\mathbf{v}_1|\mathbf{v}_2|\mathbf{v}_3) = (A\mathbf{v}_1|A\mathbf{v}_2|A\mathbf{v}_3) = (\mathbf{v}_1|3\mathbf{v}_2|-2\mathbf{v}_3) \\ &= (\mathbf{v}_1|\mathbf{v}_2|\mathbf{v}_3)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -2 \end{pmatrix} = MD. \end{aligned}$$

Columns of M are lin indep $\implies$ M is invertible.

$$\therefore M^{-1}AM = D \text{ or } A = MDM^{-1}.$$

Diagonalisation

Theorem (Diagonalisation)

Suppose $n \times n$-matrix A has a basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ of e-vectors with corresponding e-values $\lambda_1, \ldots, \lambda_n$. Let $M = (\mathbf{v}_1|\mathbf{v}_2|\ldots|\mathbf{v}_n)$ & $D = (\lambda_1\mathbf{e}_1|\ldots|\lambda_n\mathbf{e}_n)$ be the diagonal matrix with i-th diagonal entry λ_i. Then

$$M^{-1}AM = D.$$

Conversely if $M^{-1}AM = D$ with D diagonal then

- *the columns of M are a basis of e-vectors of A &*
- *the diagonal entries of D give the corresponding e-values.*

Definition

A square matrix A is **diagonalisable** if there exists an invertible matrix M and diagonal matrix D such that $M^{-1}AM = D$.

Diagonalisation: example

Example

Diagonalise $A = \begin{pmatrix} 3 & -2 \\ -1 & 2 \end{pmatrix}$ i.e. Find an invertible matrix M & diagonal matrix D, so that $M^{-1}AM = D$.

Soln We've already seen $\begin{pmatrix} 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ are e-vectors with e-values

Diagonalisation: another example

Example

Show $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is not diagonalisable by showing it does not have a basis of e-vectors.

Solution

Example cont'd

Powers of a Matrix

Suppose

$$M^{-1}AM = D \text{ or } A = MDM^{-1}.$$

Then

$$A^n = \left(MDM^{-1}\right)^n = \left(MDM^{-1}\right)\left(MDM^{-1}\right)\cdots\left(MDM^{-1}\right) = MD^nM^{-1}.$$

Easy to compute if D is diagonal!

Example

Let $A = \begin{pmatrix} 3 & -2 \\ -1 & 2 \end{pmatrix}$. Find A^{100} and A^n.

Solution

We use the diagonalisation $M^{-1}AM = D$ *where*
$M = \begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix}, D = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}.$

Example cont'd: powers of a matrix

Solution (Continued)

Example cont'd: powers of a matrix

Solution (Continued)

Why diagonal matrices are good: decoupled equations

Consider diagonal $D = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}$. Eqn $D\mathbf{x} = \mathbf{b}$ is really easy to solve $\because$ it's

$$\begin{aligned} d_1 x_1 \qquad\qquad &= b_1 \\ d_2 x_2 &= b_2 \end{aligned}$$

The 1st eqn only involves x_1 whilst the 2nd only involves x_2.
$\therefore$ can solve the eqns *separately* & we call these eqns **decoupled**.

Upshot

- row-echelon form $\Longrightarrow$ easy to solve, but
- diagonal form $\Longrightarrow$ REALLY, REALLY easy to solve.

Remark The same is true of differential equations! as we will see on the next slide.

Decoupled ODEs

Suppose that a population of hobbits at time t is $x(t)$ & the population of orcs is $y(t)$. If they live separately, say in the Shire & in Mordor, the populations grow independently according to a DE like

Example

$$\begin{cases} \dfrac{dx}{dt} = 3x \\ \dfrac{dy}{dt} = \qquad 2y \end{cases}.$$

Soln These are decoupled & we can solve for x, y *separately*

$$x(t) = \alpha e^{3t}, \quad y(t) = \beta e^{2t}.$$

Matrix form for (de)coupled ODEs

Let's now put the hobbits and orcs together in New Zealand. Then they evolve according to a coupled ODE like

Example

$$\begin{cases} \dfrac{dx}{dt} = 3x - 2y \\ \dfrac{dy}{dt} = -x + 2y \end{cases}.$$

We rewrite in matrix form. Let $\mathbf{y} = \begin{pmatrix} x \\ y \end{pmatrix}$ and $A = \begin{pmatrix} 3 & -2 \\ -1 & 2 \end{pmatrix}$, so

$$\mathbf{y}' = \begin{pmatrix} \dfrac{dx}{dt} \\ \dfrac{dy}{dt} \end{pmatrix} = \begin{pmatrix} 3x - 2y \\ -x + 2y \end{pmatrix} = \begin{pmatrix} 3 & -2 \\ -1 & 2 \end{pmatrix} \mathbf{y} \Longrightarrow \mathbf{y}' = A\mathbf{y}.$$

The fact that the ODE is coupled corresponds to the fact that A is not diagonal.

Solving ODEs by decoupling

Consider diagonalised matrix $A = MDM^{-1}$ & ODE $\frac{d\mathbf{y}}{dt} = A\mathbf{y}$.
Change variables to $\mathbf{z} = M^{-1}\mathbf{y}$ & use

Proposition

Given a constant matrix C we have $\frac{d}{dt}(C\mathbf{y}) = C\frac{d\mathbf{y}}{dt}$.

Since $M^{-1}A = DM^{-1}$ we have

$$\frac{d\mathbf{z}}{dt} = \frac{d}{dt}\left(M^{-1}\mathbf{y}\right) = M^{-1}\frac{d\mathbf{y}}{dt} = M^{-1}A\mathbf{y} = DM^{-1}\mathbf{y} = D\mathbf{z}$$

If $\lambda_1, \dots, \lambda_n$ are the diagonal entries of D, then this *decoupled* ODE in $\mathbf{z}$ can be solved as in the hobbits in the Shire/ orcs in Mordor example

$$\mathbf{z} = \begin{pmatrix} \alpha_1 e^{\lambda_1 t} \\ \vdots \\ \alpha_n e^{\lambda_n t} \end{pmatrix} \implies \mathbf{y} = M \begin{pmatrix} \alpha_1 e^{\lambda_1 t} \\ \vdots \\ \alpha_n e^{\lambda_n t} \end{pmatrix}$$

General solution to $\frac{d\mathbf{y}}{dt} = A\mathbf{y}$: explicit formula

If $M = (\mathbf{v}_1 | \dots | \mathbf{v}_n)$, we can multipy matrices to get

Theorem

Let A be an $n \times n$ matrix with a basis of e-vectors $\mathbf{v}_1, \dots, \mathbf{v}_n$ & corresponding e-values $\lambda_1, \dots, \lambda_n$. Then the general solution to $\mathbf{y}' = A\mathbf{y}$ is

$$\mathbf{y}(t) = \alpha_1 e^{\lambda_1 t}\mathbf{v}_1 + \cdots + \alpha_n e^{\lambda_n t}\mathbf{v}_n$$

for arbitrary constants $\alpha_1, \dots, \alpha_n \in \mathbb{R}$.

Note It's easy to check the expression above does give a solution.

Back to hobbits and orcs

Example

Recall the hobbit/ orc population in NZ is governed by $\frac{d\mathbf{y}}{dt} = A\mathbf{y}$ where $A = \begin{pmatrix} 3 & -2 \\ -1 & 2 \end{pmatrix}$.

1. Find the general soln to the ODE.
2. Find the population as a function of time if the initial population consisted of 4000 hobbits and 1000 orcs.

Solution

We found e-vectors for A: $\begin{pmatrix} 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ *with coresponding e-values* $4, 1$.

Hobbit/orc example cont'd

Alternate method via change of variable & decoupling

Recall $D = M^{-1}AM$ where $M = \begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix}, D = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}$.

Change variables to $\mathbf{z} = M^{-1}\mathbf{y}$. Slide 32 gives decoupled ODE $\frac{d\mathbf{z}}{dt} = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{z}$ i.e.

$$\frac{dz_1}{dt} = 4z_1, \quad \frac{dz_2}{dt} = z_2.$$

General soln $\mathbf{z} = \begin{pmatrix} z_1(t) \\ z_2(t) \end{pmatrix} = \begin{pmatrix} \alpha_1 e^{4t} \\ \alpha_2 e^{t} \end{pmatrix}, \quad \alpha_1, \alpha_2 \in \mathbb{R}$.

To find integration constants α_1, α_2, we use the initial value for $\mathbf{z}$ i.e. $\mathbf{z}(0) = M^{-1}\mathbf{y}(0)$ which can be found by solving

$$M\mathbf{z}(0) = \mathbf{y}(0) = \begin{pmatrix} 4000 \\ 1000 \end{pmatrix}.$$

Alternate method cont'd

$$\left(\begin{array}{cc|c} 2 & 1 & 4000 \\ -1 & 1 & 1000 \end{array}\right) \xrightarrow{R2=R2+\frac{1}{2}R1} \left(\begin{array}{cc|c} 2 & 1 & 4000 \\ 0 & \frac{3}{2} & 3000 \end{array}\right)$$

so $z_2(0) = \frac{2}{3} \times 3000 = 2000$ &
$2z_1(0) + 1 \times 2000 = 4000 \implies z_1(0) = 1000.$
Thus

$$\begin{pmatrix}\alpha_1 \\ \alpha_2\end{pmatrix} = \mathbf{z}(0) = \begin{pmatrix}z_1(0) \\ z_2(0)\end{pmatrix} = \begin{pmatrix}1000 \\ 2000\end{pmatrix} \implies \mathbf{z} = \begin{pmatrix}1000e^{4t} \\ 2000e^{t}\end{pmatrix}.$$

We go back to original variables

$$\mathbf{y} = M\mathbf{z} = \begin{pmatrix}2 & 1 \\ -1 & 1\end{pmatrix}\begin{pmatrix}1000e^{4t} \\ 2000e^{t}\end{pmatrix} = \begin{pmatrix}2000e^{4t} + 2000e^{t} \\ -1000e^{4t} + 2000e^{t}\end{pmatrix}.$$

Solving 2nd order ODEs using systems of 1st order ODEs

Example

Solve $y'' + 4y' - 5y = 0$ by converting this ODE into a system of first order differential equations.

Solution

The trick is to let

$$y_1 = y \quad \textit{and} \quad y_2 = y_1' = y',$$

and $\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$. *We have* $y'' = 5y - 4y' = 5y_1 - 4y_2$ *so*

$$\mathbf{y}' = \begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} y_2 \\ 5y_1 - 4y_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 5 & -4 \end{pmatrix} \mathbf{y}.$$

Solution (Continued)

E-values of $A = \begin{pmatrix} 0 & 1 \\ 5 & -4 \end{pmatrix}$ *are the solns to the characteristic eqn*

$$\begin{aligned} \det(A - \lambda I) &= 0 \\ \det \begin{pmatrix} -\lambda & 1 \\ 5 & -4-\lambda \end{pmatrix} &= 0 \\ -\lambda(-4-\lambda) - 5 &= 0 \\ \lambda^2 + 4\lambda - 5 &= 0 \\ (\lambda - 1)(\lambda + 5) &= 0 \\ \lambda = 1, -5. \end{aligned}$$

Solution (Continued)

E-vectors $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ *of* A *for e-value* 1 *are non-zero solns to*

$$(A - I)\mathbf{v} = 0 \quad i.e. \quad \begin{pmatrix} -1 & 1 \\ 5 & -5 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Hence $-v_1 + v_2 = 0$. *Pick e-vector* $\mathbf{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Sim, solve $(A + 5I)\mathbf{v} = 0$ *to get an e-vector for* $\lambda = -5$.

$$\begin{pmatrix} 5 & 1 \\ 5 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad i.e. \;\; 5v_1 + v_2 = 0.$$

Pick e-vector $\mathbf{v} = \begin{pmatrix} 1 \\ -5 \end{pmatrix}$ *for* $\lambda = -5$. *General soln is*

$$\mathbf{y} = \alpha \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t + \beta \begin{pmatrix} 1 \\ -5 \end{pmatrix} e^{-5t}.$$

Hence $y = y_1 = \alpha e^t + \beta e^{-5t}$ *for constants* α, β.

Initial Value Problem

Example

Solve the IVP $y'' + 4y' - 5y = 0, y(0) = 1, y'(0) = -5$.

Solution

Use gen soln $\mathbf{y} = \alpha \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t + \beta \begin{pmatrix} 1 \\ -5 \end{pmatrix} e^{-5t}$. *To find* α, β *note*

$$\begin{pmatrix} 1 \\ -5 \end{pmatrix} = \begin{pmatrix} y(0) \\ y'(0) \end{pmatrix} = \mathbf{y}(0) = \alpha \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \beta \begin{pmatrix} 1 \\ -5 \end{pmatrix}$$

You can now solve by Gaussian elimination, or better still note $\alpha = 0, \beta = 1$ *must be the soln! Hence*

$$\mathbf{y} = \begin{pmatrix} 1 \\ -5 \end{pmatrix} e^{-5t} \implies y = y_1 = e^{-5t}.$$