MATH1231 Algebra, 2017
Chapter 7: Linear maps

A/Prof. Daniel Chan

School of Mathematics and Statistics
University of New South Wales

danielc@unsw.edu.au

Chapter overview

Chapter overview

- $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ defined by $f\binom{x}{y} = 2x - 3y$ is example of a linear function, $g\binom{x}{y} = x^2 - 5y$ is not.

Chapter overview

- $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ defined by $f\binom{x}{y} = 2x - 3y$ is example of a linear function, $g\binom{x}{y} = x^2 - 5y$ is not.
- In this chapter, study more generally **linear transformations** $T : \mathbb{R}^m \to \mathbb{R}^n$.

Chapter overview

- $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ defined by $f\binom{x}{y} = 2x - 3y$ is example of a linear function, $g\binom{x}{y} = x^2 - 5y$ is not.
- In this chapter, study more generally **linear transformations** $T : \mathbb{R}^m \to \mathbb{R}^n$.
- Even more gen, study linear $T : V \to W$ where $V, W =$ vector spaces $/\mathbb{F}$.

Chapter overview

- $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ defined by $f\binom{x}{y} = 2x - 3y$ is example of a linear function, $g\binom{x}{y} = x^2 - 5y$ is not.
- In this chapter, study more generally **linear transformations** $T : \mathbb{R}^m \to \mathbb{R}^n$.
- Even more gen, study linear $T : V \to W$ where $V, W =$ vector spaces $/\mathbb{F}$. Recall V is the **domain** of T & W the **codomain** of T.

Chapter overview

- $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ defined by $f\binom{x}{y} = 2x - 3y$ is example of a linear function, $g\binom{x}{y} = x^2 - 5y$ is not.
- In this chapter, study more generally **linear transformations** $T : \mathbb{R}^m \to \mathbb{R}^n$.
- Even more gen, study linear $T : V \to W$ where $V, W =$ vector spaces $/\mathbb{F}$. Recall V is the **domain** of T & W the **codomain** of T.
- We'll generalise systems of linear equations $A\mathbf{x} = \mathbf{b}$ to "linear equations" of form $T\mathbf{x} = \mathbf{b}$ where $\mathbf{b} \in W, \mathbf{x} \in V$.

Chapter overview

- $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ defined by $f\binom{x}{y} = 2x - 3y$ is example of a linear function, $g\binom{x}{y} = x^2 - 5y$ is not.
- In this chapter, study more generally **linear transformations** $T : \mathbb{R}^m \to \mathbb{R}^n$.
- Even more gen, study linear $T : V \to W$ where $V, W =$ vector spaces $/\mathbb{F}$. Recall V is the **domain** of T & W the **codomain** of T.
- We'll generalise systems of linear equations $A\mathbf{x} = \mathbf{b}$ to "linear equations" of form $T\mathbf{x} = \mathbf{b}$ where $\mathbf{b} \in W, \mathbf{x} \in V$.

Often abbrev $T(\mathbf{x}) = T\mathbf{x}$.

Addition Condition

Addition Condition

To define linear map, first consider

Addition Condition

To define linear map, first consider

Addition Condition.

We say $T : V \longrightarrow W$ satisfies the **addition condition**, if

To define linear map, first consider

Addition Condition.

We say $T : V \longrightarrow W$ satisfies the **addition condition**, if

$$T(\mathbf{v} + \mathbf{v}') = T(\mathbf{v}) + T(\mathbf{v}') \text{ for all } \mathbf{v}, \mathbf{v}' \in V.$$

Addition Condition

To define linear map, first consider

Addition Condition.

We say $T : V \longrightarrow W$ satisfies the **addition condition**, if

$$T(\mathbf{v} + \mathbf{v}') = T(\mathbf{v}) + T(\mathbf{v}') \text{ for all } \mathbf{v}, \mathbf{v}' \in V.$$

E.g. $T : \mathbb{R}^2 \longrightarrow \mathbb{R}$ defined by $T\binom{x}{y} = 2x - 3y$ satisfies the addn condn since given $\binom{x}{y}, \binom{x'}{y'} \in \mathbb{R}^2$

Addition Condition

To define linear map, first consider

Addition Condition.

We say $T : V \longrightarrow W$ satisfies the **addition condition**, if

$$T(\mathbf{v} + \mathbf{v}') = T(\mathbf{v}) + T(\mathbf{v}') \text{ for all } \mathbf{v}, \mathbf{v}' \in V.$$

E.g. $T : \mathbb{R}^2 \longrightarrow \mathbb{R}$ defined by $T\binom{x}{y} = 2x - 3y$ satisfies the addn condn since given $\binom{x}{y}, \binom{x'}{y'} \in \mathbb{R}^2$

$$T\left(\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} x' \\ y' \end{pmatrix}\right) =$$

To define linear map, first consider

Addition Condition.

We say $T : V \longrightarrow W$ satisfies the **addition condition**, if

$$T(\mathbf{v} + \mathbf{v}') = T(\mathbf{v}) + T(\mathbf{v}') \text{ for all } \mathbf{v}, \mathbf{v}' \in V.$$

E.g. $T : \mathbb{R}^2 \longrightarrow \mathbb{R}$ defined by $T\binom{x}{y} = 2x - 3y$ satisfies the addn condn since given $\binom{x}{y}, \binom{x'}{y'} \in \mathbb{R}^2$

$$T\left(\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} x' \\ y' \end{pmatrix}\right) =$$

$$T\begin{pmatrix} x \\ y \end{pmatrix} + T\begin{pmatrix} x' \\ y' \end{pmatrix} =$$

Addition Condition

To define linear map, first consider

Addition Condition.

We say $T : V \longrightarrow W$ satisfies the **addition condition**, if

$$T(\mathbf{v} + \mathbf{v}') = T(\mathbf{v}) + T(\mathbf{v}') \text{ for all } \mathbf{v}, \mathbf{v}' \in V.$$

E.g. $T : \mathbb{R}^2 \longrightarrow \mathbb{R}$ defined by $T\binom{x}{y} = 2x - 3y$ satisfies the addn condn since given $\binom{x}{y}, \binom{x'}{y'} \in \mathbb{R}^2$

$$T\left(\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} x' \\ y' \end{pmatrix}\right) =$$

$$T\begin{pmatrix} x \\ y \end{pmatrix} + T\begin{pmatrix} x' \\ y' \end{pmatrix} =$$

Warning The $+$ on the two sides of the equation are different!

Scalar Multiplication Condition

Scalar Multiplication Condition.

We say $T : V \longrightarrow W$ satisfies the **scalar multiplication condition**, if

Scalar Multiplication Condition.

We say $T : V \longrightarrow W$ satisfies the **scalar multiplication condition**, if

$$T(\lambda \mathbf{v}) = \lambda T(\mathbf{v}) \text{ for all } \lambda \in \mathbb{F} \text{ and } \mathbf{v} \in V.$$

Scalar Multiplication Condition.

We say $T : V \longrightarrow W$ satisfies the **scalar multiplication condition**, if

$$T(\lambda \mathbf{v}) = \lambda T(\mathbf{v}) \text{ for all } \lambda \in \mathbb{F} \text{ and } \mathbf{v} \in V.$$

E.g. $T : \mathbb{R}^2 \longrightarrow \mathbb{R}$ defined by $T\binom{x}{y} = 2x - 3y$ satisfies the scalar multn condn since given $\binom{x}{y} \in \mathbb{R}^2, \lambda \in \mathbb{R}$

Scalar Multiplication Condition

Scalar Multiplication Condition.

We say $T : V \longrightarrow W$ satisfies the **scalar multiplication condition**, if

$$T(\lambda \mathbf{v}) = \lambda T(\mathbf{v}) \text{ for all } \lambda \in \mathbb{F} \text{ and } \mathbf{v} \in V.$$

E.g. $T : \mathbb{R}^2 \longrightarrow \mathbb{R}$ defined by $T\binom{x}{y} = 2x - 3y$ satisfies the scalar multn condn since given $\binom{x}{y} \in \mathbb{R}^2, \lambda \in \mathbb{R}$

$$T\left(\lambda \begin{pmatrix} x \\ y \end{pmatrix}\right) =$$

Scalar Multiplication Condition.

We say $T : V \longrightarrow W$ satisfies the **scalar multiplication condition**, if

$$T(\lambda \mathbf{v}) = \lambda T(\mathbf{v}) \text{ for all } \lambda \in \mathbb{F} \text{ and } \mathbf{v} \in V.$$

E.g. $T : \mathbb{R}^2 \longrightarrow \mathbb{R}$ defined by $T\binom{x}{y} = 2x - 3y$ satisfies the scalar multn condn since given $\binom{x}{y} \in \mathbb{R}^2, \lambda \in \mathbb{R}$

$$T\left(\lambda\begin{pmatrix} x \\ y \end{pmatrix}\right) =$$

$$\lambda T\begin{pmatrix} x \\ y \end{pmatrix} =$$

Scalar Multiplication Condition

Scalar Multiplication Condition.

We say $T : V \longrightarrow W$ satisfies the **scalar multiplication condition**, if

$$T(\lambda \mathbf{v}) = \lambda T(\mathbf{v}) \text{ for all } \lambda \in \mathbb{F} \text{ and } \mathbf{v} \in V.$$

E.g. $T : \mathbb{R}^2 \longrightarrow \mathbb{R}$ defined by $T\binom{x}{y} = 2x - 3y$ satisfies the scalar multn condn since given $\binom{x}{y} \in \mathbb{R}^2, \lambda \in \mathbb{R}$

$$T\left(\lambda \begin{pmatrix} x \\ y \end{pmatrix}\right) =$$

$$\lambda T \begin{pmatrix} x \\ y \end{pmatrix} =$$

Warning The scalar multn on the two sides of the eqn are different!

Linear Transformation

Definition

Let $V, W =$ vector spaces $/\mathbb{F}$.

Linear Transformation

Definition

Let $V, W =$ vector spaces $/\mathbb{F}$. A function $T : V \to W$ is called a **linear map** or a **linear transformation** if following both hold.

Linear Transformation

Definition

Let $V, W =$ vector spaces $/\mathbb{F}$. A function $T : V \to W$ is called a **linear map** or a **linear transformation** if following both hold.

Addition Condition.

$$T(\mathbf{v} + \mathbf{v}') = T(\mathbf{v}) + T(\mathbf{v}') \text{ for all } \mathbf{v}, \mathbf{v}' \in V, \text{ and}$$

Linear Transformation

Definition

Let $V, W =$ vector spaces $/\mathbb{F}$. A function $T : V \to W$ is called a **linear map** or a **linear transformation** if following both hold.

Addition Condition.

$$T(\mathbf{v} + \mathbf{v}') = T(\mathbf{v}) + T(\mathbf{v}') \text{ for all } \mathbf{v}, \mathbf{v}' \in V, \text{ and}$$

Scalar Multiplication Condition.

$$T(\lambda \mathbf{v}) = \lambda T(\mathbf{v}) \text{ for all } \lambda \in \mathbb{F} \text{ and } \mathbf{v} \in V.$$

Linear Transformation

Definition

Let $V, W =$ vector spaces $/\mathbb{F}$. A function $T : V \to W$ is called a **linear map** or a **linear transformation** if following both hold.

Addition Condition.

$$T(\mathbf{v} + \mathbf{v}') = T(\mathbf{v}) + T(\mathbf{v}') \text{ for all } \mathbf{v}, \mathbf{v}' \in V, \text{ and}$$

Scalar Multiplication Condition.

$$T(\lambda \mathbf{v}) = \lambda T(\mathbf{v}) \text{ for all } \lambda \in \mathbb{F} \text{ and } \mathbf{v} \in V.$$

E.g. $T : \mathbb{R}^2 \longrightarrow \mathbb{R}$ defined by $T\binom{x}{y} = 2x - 3y$ is linear.

Sample question: showing a function is linear.

Example

Show that the function $T : \mathbb{R}^3 \to \mathbb{R}^2$ defined by

$$T(\mathbf{x}) = \begin{pmatrix} 4x_2 - 3x_3 \\ x_1 + 2x_2 \end{pmatrix} \quad \text{for} \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3$$

is a linear map.

Solution

Solution (Continued)

Solution (Continued)

Linear maps preserve zero.

Proposition.

If $T : V \longrightarrow W$ *is a linear map, then* $T(\mathbf{0}) = \mathbf{0}$.

Linear maps preserve zero.

Proposition.

If $T : V \longrightarrow W$ is a linear map, then $T(\mathbf{0}) = \mathbf{0}$.

Proof. $T(\mathbf{0}) = T(0\mathbf{0}) = 0T(\mathbf{0}) = \mathbf{0}$.

Linear maps preserve zero.

Proposition.

If $T : V \longrightarrow W$ is a linear map, then $T(\mathbf{0}) = \mathbf{0}$.

Proof. $T(\mathbf{0}) = T(0\mathbf{0}) = 0T(\mathbf{0}) = \mathbf{0}$.

Example

Show that the function $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ x_2 - 2 \\ x_1 \end{pmatrix}$ is not linear.

Soln

Another non-linear example

Another non-linear example

Example

Show that the function $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$T \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ x_2^2 \end{pmatrix}$$

is not linear.

Solution

Checking $T(\mathbf{0}) = \mathbf{0}$ here tells you nothing about linearity.
Suffice check that $T(\lambda \mathbf{v}) = \lambda T \mathbf{v}$ fails for single choice of pair $\lambda, \mathbf{v}$.

Alternate characterisation of linearity

Alternate characterisation of linearity

We can combine the addn condn & scalar multn condn into one!

Alternate characterisation of linearity

We can combine the addn condn & scalar multn condn into one!

Theorem

The function $T : V \to W$ is a linear map iff for all $\lambda \in \mathbb{F}$ and $\mathbf{v}_1, \mathbf{v}_2 \in V$

$$T(\lambda \mathbf{v}_1 + \mathbf{v}_2) = \lambda T(\mathbf{v}_1) + T(\mathbf{v}_2).$$

Alternate characterisation of linearity

We can combine the addn condn & scalar multn condn into one!

Theorem

The function $T : V \to W$ *is a linear map iff for all* $\lambda \in \mathbb{F}$ *and* $\mathbf{v}_1, \mathbf{v}_2 \in V$

$$T(\lambda \mathbf{v}_1 + \mathbf{v}_2) = \lambda T(\mathbf{v}_1) + T(\mathbf{v}_2).$$

Remark This means that a linear map T has the special property that it sends the line $\mathbf{x} = \mathbf{a} + \lambda \mathbf{v}$ to the line $\mathbf{x} = T\mathbf{a} + \lambda T\mathbf{v}$ or point $T\mathbf{a}$ if $T\mathbf{v} = \mathbf{0}$.

Alternate characterisation of linearity

We can combine the addn condn & scalar multn condn into one!

Theorem

The function $T : V \to W$ *is a linear map iff for all* $\lambda \in \mathbb{F}$ *and* $\mathbf{v}_1, \mathbf{v}_2 \in V$

$$T(\lambda \mathbf{v}_1 + \mathbf{v}_2) = \lambda T(\mathbf{v}_1) + T(\mathbf{v}_2).$$

Remark This means that a linear map T has the special property that it sends the line $\mathbf{x} = \mathbf{a} + \lambda \mathbf{v}$ to the line $\mathbf{x} = T\mathbf{a} + \lambda T\mathbf{v}$ or point $T\mathbf{a}$ if $T\mathbf{v} = \mathbf{0}$.
E.g. Differentation is a linear map.

We can combine the addn condn & scalar multn condn into one!

Theorem

The function $T : V \to W$ *is a linear map iff for all* $\lambda \in \mathbb{F}$ *and* $\mathbf{v}_1, \mathbf{v}_2 \in V$

$$T(\lambda \mathbf{v}_1 + \mathbf{v}_2) = \lambda T(\mathbf{v}_1) + T(\mathbf{v}_2).$$

Remark This means that a linear map T has the special property that it sends the line $\mathbf{x} = \mathbf{a} + \lambda \mathbf{v}$ to the line $\mathbf{x} = T\mathbf{a} + \lambda T\mathbf{v}$ or point $T\mathbf{a}$ if $T\mathbf{v} = \mathbf{0}$.
E.g. Differentation is a linear map. More precisely, we define $T : \mathbb{P} \longrightarrow \mathbb{P}$ by $Tp = \frac{dp}{dx}$. Then for $p, q \in \mathbb{P}, \lambda \in \mathbb{R}$

Alternate characterisation of linearity

We can combine the addn condn & scalar multn condn into one!

Theorem

The function $T : V \to W$ is a linear map iff for all $\lambda \in \mathbb{F}$ and $\mathbf{v}_1, \mathbf{v}_2 \in V$

$$T(\lambda \mathbf{v}_1 + \mathbf{v}_2) = \lambda T(\mathbf{v}_1) + T(\mathbf{v}_2).$$

Remark This means that a linear map T has the special property that it sends the line $\mathbf{x} = \mathbf{a} + \lambda \mathbf{v}$ to the line $\mathbf{x} = T\mathbf{a} + \lambda T\mathbf{v}$ or point $T\mathbf{a}$ if $T\mathbf{v} = \mathbf{0}$.
E.g. Differentation is a linear map. More precisely, we define $T : \mathbb{P} \longrightarrow \mathbb{P}$ by $Tp = \frac{dp}{dx}$. Then for $p, q \in \mathbb{P}, \lambda \in \mathbb{R}$

$$T(\lambda p + q) =$$

Theorem

If $T : V \to W$ *is linear map,* $\mathbf{v}_1, \ldots, \mathbf{v}_n \in V$ *&* $\lambda_1, \ldots, \lambda_n$ *are scalars, then*

$$T(\lambda_1 \mathbf{v}_1 + \cdots + \lambda_n \mathbf{v}_n) = \lambda_1 T(\mathbf{v}_1) + \cdots + \lambda_n T(\mathbf{v}_n).$$

Theorem

If $T : V \to W$ is linear map, $\mathbf{v}_1, \ldots, \mathbf{v}_n \in V$ & $\lambda_1, \ldots, \lambda_n$ are scalars, then

$$T(\lambda_1 \mathbf{v}_1 + \cdots + \lambda_n \mathbf{v}_n) = \lambda_1 T(\mathbf{v}_1) + \cdots + \lambda_n T(\mathbf{v}_n).$$

Example

If $T : \mathbb{R}^2 \to \mathbb{R}^2$ a function such that

$$T\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \; T\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \; T\begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}.$$

Show that T is not linear.

Solution

Solution (Continued)

Example

Given that T is a linear map and

$$T\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix},\ T\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}2\\-1\end{pmatrix},\ T\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}0\\3\end{pmatrix}.$$

Find $T\begin{pmatrix}x\\y\\z\end{pmatrix}$.

Solution

Linear maps are determined by the values on a spanning set

Previous eg illustrates the following general result.

Linear maps are determined by the values on a spanning set

Previous eg illustrates the following general result.

Theorem

Let $T : V \longrightarrow W$ be linear & $V = span(\mathbf{v}_1, \ldots, \mathbf{v}_m)$. Then T is completely determined by the m values $T\mathbf{v}_1, \ldots, T\mathbf{v}_m$.

Linear maps are determined by the values on a spanning set

Previous eg illustrates the following general result.

Theorem

Let $T : V \longrightarrow W$ be linear & $V = span(\mathbf{v}_1, \ldots, \mathbf{v}_m)$. Then T is completely determined by the m values $T\mathbf{v}_1, \ldots, T\mathbf{v}_m$.

Compare with the following:

Linear maps are determined by the values on a spanning set

Previous eg illustrates the following general result.

Theorem

Let $T : V \longrightarrow W$ be linear & $V = span(\mathbf{v}_1, \ldots, \mathbf{v}_m)$. Then T is completely determined by the m values $T\mathbf{v}_1, \ldots, T\mathbf{v}_m$.

Compare with the following:

An affine linear function $f(x) = mx + b$ is determined by two of its values $f(x_1), f(x_2)$, since

Linear maps are determined by the values on a spanning set

Previous eg illustrates the following general result.

Theorem

Let $T : V \longrightarrow W$ be linear & $V = span(\mathbf{v}_1, \ldots, \mathbf{v}_m)$. Then T is completely determined by the m values $T\mathbf{v}_1, \ldots, T\mathbf{v}_m$.

Compare with the following:

An affine linear function $f(x) = mx + b$ is determined by two of its values $f(x_1), f(x_2)$, since its graph is a line which is determined by two points.

Matrices define Linear Maps

Theorem

For each $m \times n$ matrix A, the function $T_A : \mathbb{R}^n \to \mathbb{R}^m$, defined by

$$T_A(\mathbf{x}) = A\mathbf{x} \qquad \text{for} \quad \mathbf{x} \in \mathbb{R}^n,$$

is a linear map called the **associated linear map***.*

Proof.

□

Example of reflection

Example

Let $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, describe the associated linear map T_A geometrically as a mapping $\mathbb{R}^2 \longrightarrow \mathbb{R}^2$.

Solution

Matrix Representation Theorem

Conversely, given linear $T : \mathbb{R}^n \to \mathbb{R}^m$, we can find an $m \times n$ matrix A such that $T(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^n$.

Matrix Representation Theorem

Conversely, given linear $T : \mathbb{R}^n \to \mathbb{R}^m$, we can find an $m \times n$ matrix A such that $T(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^n$. In this case, we say A is a matrix **representing** T.

Matrix Representation Theorem

Conversely, given linear $T : \mathbb{R}^n \to \mathbb{R}^m$, we can find an $m \times n$ matrix A such that $T(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^n$. In this case, we say A is a matrix **representing** T.

Example

Given that $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ defined by $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + 2y \\ 2x - y \\ y \end{pmatrix}$ is linear. Find the matrix A representing T.

Matrix Representation Theorem

Conversely, given linear $T : \mathbb{R}^n \to \mathbb{R}^m$, we can find an $m \times n$ matrix A such that $T(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^n$. In this case, we say A is a matrix **representing** T.

Example

Given that $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ defined by $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + 2y \\ 2x - y \\ y \end{pmatrix}$ is linear. Find the matrix A representing T.

Solution

Formula for representing matrix

Formula for representing matrix

Theorem

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map and let the vectors $\mathbf{e}_j$ for $1 \leqslant j \leqslant n$ be the standard basis vectors for $\mathbb{R}^n$.

Formula for representing matrix

Theorem

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ *be a linear map and let the vectors* $\mathbf{e}_j$ *for* $1 \leqslant j \leqslant n$ *be the standard basis vectors for* $\mathbb{R}^n$*. Then the* $m \times n$ *matrix*

$$A = (T\mathbf{e}_1 | T\mathbf{e}_2 | \dots | T\mathbf{e}_n)$$

has the property that

$$T(\mathbf{x}) = A\mathbf{x} \quad \textit{for all} \quad \mathbf{x} \in \mathbb{R}^n.$$

Formula for representing matrix

Theorem

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ *be a linear map and let the vectors* $\mathbf{e}_j$ *for* $1 \leqslant j \leqslant n$ *be the standard basis vectors for* $\mathbb{R}^n$*. Then the* $m \times n$ *matrix*

$$A = (T\mathbf{e}_1 | T\mathbf{e}_2 | \ldots | T\mathbf{e}_n)$$

has the property that

$$T(\mathbf{x}) = A\mathbf{x} \quad \textit{for all} \quad \mathbf{x} \in \mathbb{R}^n.$$

E.g. In the example of the previous slide,

Formula for representing matrix

Theorem

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map and let the vectors $\mathbf{e}_j$ for $1 \leqslant j \leqslant n$ be the standard basis vectors for $\mathbb{R}^n$. Then the $m \times n$ matrix

$$A = (T\mathbf{e}_1 | T\mathbf{e}_2 | \dots | T\mathbf{e}_n)$$

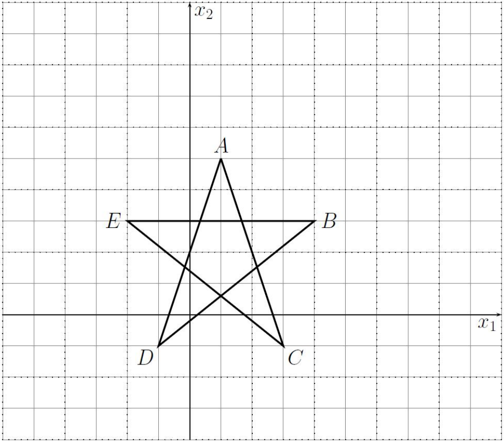
has the property that

$$T(\mathbf{x}) = A\mathbf{x} \quad \text{for all} \quad \mathbf{x} \in \mathbb{R}^n.$$

E.g. In the example of the previous slide,

$T\mathbf{e}_1 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$, $T\mathbf{e}_2 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$ so the representing matrix is

Stretching and Compressing



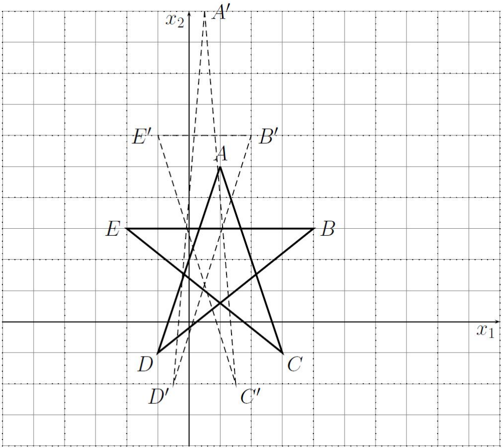
A 5-point star with vertices
$A(1,5)$, $B(4,3)$, $C(3,-1)$, $D(-1,-1)$ and $E(-2,3)$.

Example

Find and draw the image of the 5-point star under the linear map T_M defined by the matrix $M = \begin{pmatrix} 0.5 & 0 \\ 0 & 2 \end{pmatrix}$.

Solution

Solution (Continued)



Rotation about $\mathbf{0}$ is linear

Consider the map R_α, which rotates the $\mathbb{R}^2$ plane through an angle α anticlockwise about the origin.

Rotation about $\mathbf{0}$ is linear

Consider the map R_α, which rotates the $\mathbb{R}^2$ plane through an angle α anticlockwise about the origin.

One can show geometrically that R_α is a linear map see Section 7.3. example 3.

Rotation about $\mathbf{0}$ is linear

Consider the map R_α, which rotates the $\mathbb{R}^2$ plane through an angle α anticlockwise about the origin.

One can show geometrically that R_α is a linear map see Section 7.3. example 3.

Example

Find the matrix A representing R_α.

Rotation about $\mathbf{0}$ is linear

Consider the map R_α, which rotates the $\mathbb{R}^2$ plane through an angle α anticlockwise about the origin.

One can show geometrically that R_α is a linear map see Section 7.3. example 3.

Example

Find the matrix A representing R_α.

Solution

Projection onto **b** is linear

Projection onto $\mathbf{b}$ is linear

Recall given given $\mathbf{b} \in \mathbb{R}^n$ we have a projection map $\text{proj}_{\mathbf{b}} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ which sends $\mathbf{x} \mapsto \text{proj}_{\mathbf{b}}\mathbf{x}$.

Projection onto $\mathbf{b}$ is linear

Recall given given $\mathbf{b} \in \mathbb{R}^n$ we have a projection map $\text{proj}_{\mathbf{b}} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ which sends $\mathbf{x} \mapsto \text{proj}_{\mathbf{b}}\mathbf{x}$.

Proposition

$$\text{proj}_{\mathbf{b}}\mathbf{x} = \frac{1}{|\mathbf{b}|^2}\mathbf{b}\mathbf{b}^T\mathbf{x}$$

Projection onto $\mathbf{b}$ is linear

Recall given given $\mathbf{b} \in \mathbb{R}^n$ we have a projection map $\text{proj}_{\mathbf{b}} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ which sends $\mathbf{x} \mapsto \text{proj}_{\mathbf{b}}\mathbf{x}$.

Proposition

$$\text{proj}_{\mathbf{b}}\mathbf{x} = \frac{1}{|\mathbf{b}|^2}\mathbf{b}\mathbf{b}^T\mathbf{x}$$

Hence $\text{proj}_{\mathbf{b}}$ is linear being the linear map associated to the matrix

Projection onto $\mathbf{b}$ is linear

Recall given given $\mathbf{b} \in \mathbb{R}^n$ we have a projection map $\text{proj}_{\mathbf{b}} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ which sends $\mathbf{x} \mapsto \text{proj}_{\mathbf{b}}\mathbf{x}$.

Proposition

$$\text{proj}_{\mathbf{b}}\mathbf{x} = \frac{1}{|\mathbf{b}|^2}\mathbf{b}\mathbf{b}^T\mathbf{x}$$

Hence $\text{proj}_{\mathbf{b}}$ is linear being the linear map associated to the matrix

$$A = \frac{1}{|\mathbf{b}|^2}\mathbf{b}\mathbf{b}^T.$$

Projection onto $\mathbf{b}$ is linear

Recall given given $\mathbf{b} \in \mathbb{R}^n$ we have a projection map $\text{proj}_{\mathbf{b}} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ which sends $\mathbf{x} \mapsto \text{proj}_{\mathbf{b}}\mathbf{x}$.

Proposition

$$\text{proj}_{\mathbf{b}}\mathbf{x} = \frac{1}{|\mathbf{b}|^2}\mathbf{b}\mathbf{b}^T\mathbf{x}$$

Hence $\text{proj}_{\mathbf{b}}$ is linear being the linear map associated to the matrix

$$A = \frac{1}{|\mathbf{b}|^2}\mathbf{b}\mathbf{b}^T.$$

Proof.

Note

$$A\mathbf{x} = \frac{1}{|\mathbf{b}|^2}\mathbf{b}\mathbf{b}^T\mathbf{x} = \frac{1}{|\mathbf{b}|^2}\mathbf{b}(\mathbf{b}\cdot\mathbf{x}) = \frac{\mathbf{b}\cdot\mathbf{x}}{|\mathbf{b}|^2}\mathbf{b} = \text{proj}_{\mathbf{b}}\mathbf{x}$$

Projection onto $\mathbf{b}$ is linear

Recall given given $\mathbf{b} \in \mathbb{R}^n$ we have a projection map $\text{proj}_{\mathbf{b}} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ which sends $\mathbf{x} \mapsto \text{proj}_{\mathbf{b}}\mathbf{x}$.

Proposition

$$\text{proj}_{\mathbf{b}}\mathbf{x} = \frac{1}{|\mathbf{b}|^2}\mathbf{b}\mathbf{b}^T\mathbf{x}$$

Hence $\text{proj}_{\mathbf{b}}$ is linear being the linear map associated to the matrix

$$A = \frac{1}{|\mathbf{b}|^2}\mathbf{b}\mathbf{b}^T.$$

Proof.

Note

$$A\mathbf{x} = \frac{1}{|\mathbf{b}|^2}\mathbf{b}\mathbf{b}^T\mathbf{x} = \frac{1}{|\mathbf{b}|^2}\mathbf{b}(\mathbf{b}\cdot\mathbf{x}) = \frac{\mathbf{b}\cdot\mathbf{x}}{|\mathbf{b}|^2}\mathbf{b} = \text{proj}_{\mathbf{b}}\mathbf{x}$$

from the formula for $\text{proj}_{\mathbf{b}}\mathbf{x}$ given in MATH1131 □

Sample projection

Sample projection

Example

Let $\mathbf{b} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $T = \text{proj}_{\mathbf{b}} : \mathbb{R}^2 \to \mathbb{R}^2$.

Sample projection

Example

Let $\mathbf{b} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $T = \text{proj}_{\mathbf{b}} : \mathbb{R}^2 \to \mathbb{R}^2$.

i) Find the matrix A representing T.

Sample projection

Example

Let $\mathbf{b} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $T = \text{proj}_{\mathbf{b}} : \mathbb{R}^2 \to \mathbb{R}^2$.

i) Find the matrix A representing T. ii) Check your answer by computing the linear map associated to the matrix A you found.

Solution

Kernels of linear maps

Kernels of linear maps

Let $T : V \to W$ be a linear map.

Kernels of linear maps

Let $T : V \to W$ be a linear map.

Proposition-Definition

The **kernel** of T (written $\ker(T)$ or $\ker T$) is the set,

$$\ker(T) = \{\mathbf{v} \in V | T(\mathbf{v}) = \mathbf{0}\} \subseteq V.$$

Kernels of linear maps

Let $T : V \to W$ be a linear map.

Proposition-Definition

The **kernel** of T (written $\ker(T)$ or $\ker T$) is the set,

$$\ker(T) = \{\mathbf{v} \in V | T(\mathbf{v}) = \mathbf{0}\} \subseteq V.$$

Let $A \in M_{mn}(\mathbb{R})$ & $T_A : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be the assoc linear map.

Kernels of linear maps

Let $T : V \to W$ be a linear map.

Proposition-Definition

The **kernel** of T (written $\ker(T)$ or $\ker T$) is the set,

$$\ker(T) = \{\mathbf{v} \in V | T(\mathbf{v}) = \mathbf{0}\} \subseteq V.$$

Let $A \in M_{mn}(\mathbb{R})$ & $T_A : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be the assoc linear map. We define

$$\ker A = \ker T_A = \{\mathbf{v} \in \mathbb{R}^n : A\mathbf{v} = \mathbf{0}\}.$$

Kernels of linear maps

Let $T : V \to W$ be a linear map.

Proposition-Definition

The **kernel** of T (written $\ker(T)$ or $\ker T$) is the set,

$$\ker(T) = \{\mathbf{v} \in V | T(\mathbf{v}) = \mathbf{0}\} \subseteq V.$$

Let $A \in M_{mn}(\mathbb{R})$ & $T_A : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be the assoc linear map. We define

$$\ker A = \ker T_A = \{\mathbf{v} \in \mathbb{R}^n : A\mathbf{v} = \mathbf{0}\}.$$

$\ker T$ is a subspace of V.

Kernels of linear maps

Let $T : V \to W$ be a linear map.

Proposition-Definition

The **kernel** of T (written $\ker(T)$ or $\ker T$) is the set,

$$\ker(T) = \{\mathbf{v} \in V | T(\mathbf{v}) = \mathbf{0}\} \subseteq V.$$

Let $A \in M_{mn}(\mathbb{R})$ & $T_A : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be the assoc linear map. We define

$$\ker A = \ker T_A = \{\mathbf{v} \in \mathbb{R}^n : A\mathbf{v} = \mathbf{0}\}.$$

$\ker T$ is a subspace of V.

E.g. Is $\binom{1}{2}$ in $\ker(2 \ \ -1)$?

Kernels of linear maps

Let $T : V \to W$ be a linear map.

Proposition-Definition

The **kernel** of T (written $\ker(T)$ or $\ker\ T$) is the set,

$$\ker(T) = \{\mathbf{v} \in V | T(\mathbf{v}) = \mathbf{0}\} \subseteq V.$$

Let $A \in M_{mn}(\mathbb{R})$ & $T_A : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be the assoc linear map. We define

$$\ker A = \ker T_A = \{\mathbf{v} \in \mathbb{R}^n : A\mathbf{v} = \mathbf{0}\}.$$

$\ker T$ is a subspace of V.

E.g. Is $\binom{1}{2}$ in $\ker(2 \;\; -1)$?

E.g. Consider the differentiation map $T : \mathbb{P} \longrightarrow \mathbb{P}, (Tp)(x) = p'(x)$.

Kernels of linear maps

Let $T : V \to W$ be a linear map.

Proposition-Definition

The **kernel** of T (written $\ker(T)$ or $\ker\ T$) is the set,

$$\ker(T) = \{\mathbf{v} \in V | T(\mathbf{v}) = \mathbf{0}\} \subseteq V.$$

Let $A \in M_{mn}(\mathbb{R})$ & $T_A : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be the assoc linear map. We define

$$\ker A = \ker T_A = \{\mathbf{v} \in \mathbb{R}^n : A\mathbf{v} = \mathbf{0}\}.$$

$\ker T$ is a subspace of V.

E.g. Is $\binom{1}{2}$ in $\ker(2\ \ -1)$?

E.g. Consider the differentiation map $T : \mathbb{P} \longrightarrow \mathbb{P}, (Tp)(x) = p'(x)$.
$\ker T = \{p \in \mathbb{P} | \frac{dp}{dx} = 0\} = \mathbb{P}_0$

Kernels of linear maps

Let $T : V \to W$ be a linear map.

Proposition-Definition

The **kernel** of T (written $\ker(T)$ or $\ker T$) is the set,

$$\ker(T) = \{\mathbf{v} \in V | T(\mathbf{v}) = \mathbf{0}\} \subseteq V.$$

Let $A \in M_{mn}(\mathbb{R})$ & $T_A : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be the assoc linear map. We define

$$\ker A = \ker T_A = \{\mathbf{v} \in \mathbb{R}^n : A\mathbf{v} = \mathbf{0}\}.$$

$\ker T$ is a subspace of V.

E.g. Is $\binom{1}{2}$ in $\ker(2 \ \ -1)$?

E.g. Consider the differentiation map $T : \mathbb{P} \longrightarrow \mathbb{P}, (Tp)(x) = p'(x)$.
$\ker T = \{p \in \mathbb{P} | \frac{dp}{dx} = 0\} = \mathbb{P}_0$ the subspace of all constant polynomials.

Proof that kernels are subspaces

Proof that kernels are subspaces

Let $T : V \longrightarrow W$ be a linear map. We prove that $\ker T$ is a subspace of V by checking closure axioms.

Proof that kernels are subspaces

Let $T : V \longrightarrow W$ be a linear map. We prove that $\ker T$ is a subspace of V by checking closure axioms.

Proof.

□

Images of linear maps

Images of linear maps

Let $T : V \to W$ be a linear map.

Images of linear maps

Let $T : V \to W$ be a linear map.

Proposition-Definition

The **image** of T is the set of all function values of T, that is,

$$\text{im}(T) = \{\mathbf{w} \in W : \mathbf{w} = T(\mathbf{v}) \quad \text{for some} \quad \mathbf{v} \in V\} \subseteq W.$$

Images of linear maps

Let $T : V \to W$ be a linear map.

Proposition-Definition

The **image** of T is the set of all function values of T, that is,

$$\text{im}(T) = \{\mathbf{w} \in W : \mathbf{w} = T(\mathbf{v}) \quad \text{for some} \quad \mathbf{v} \in V\} \subseteq W.$$

Let $A \in M_{mn}(\mathbb{R})$ & $T_A : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be the assoc linear map.

Images of linear maps

Let $T : V \to W$ be a linear map.

Proposition-Definition

The **image** of T is the set of all function values of T, that is,

$$\text{im}(T) = \{\mathbf{w} \in W : \mathbf{w} = T(\mathbf{v}) \quad \text{for some} \quad \mathbf{v} \in V\} \subseteq W.$$

Let $A \in M_{mn}(\mathbb{R})$ & $T_A : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be the assoc linear map. We define

$$\text{im } A = \text{im } T_A = \{\mathbf{b} \in \mathbb{R}^m : \mathbf{b} = A\mathbf{x} \quad \text{for some} \quad \mathbf{x} \in \mathbb{R}^n\} =$$

Images of linear maps

Let $T : V \to W$ be a linear map.

Proposition-Definition

The **image** of T is the set of all function values of T, that is,

$$\mathrm{im}(T) = \{\mathbf{w} \in W : \mathbf{w} = T(\mathbf{v}) \quad \text{for some} \quad \mathbf{v} \in V\} \subseteq W.$$

Let $A \in M_{mn}(\mathbb{R})$ & $T_A : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be the assoc linear map. We define

$$\mathrm{im}\ A = \mathrm{im}\ T_A = \{\mathbf{b} \in \mathbb{R}^m : \mathbf{b} = A\mathbf{x} \quad \text{for some} \quad \mathbf{x} \in \mathbb{R}^n\} = \mathrm{col}(A).$$

Images of linear maps

Let $T : V \to W$ be a linear map.

Proposition-Definition

The **image** of T is the set of all function values of T, that is,

$$\text{im}(T) = \{\mathbf{w} \in W : \mathbf{w} = T(\mathbf{v}) \quad \text{for some} \quad \mathbf{v} \in V\} \subseteq W.$$

Let $A \in M_{mn}(\mathbb{R})$ & $T_A : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be the assoc linear map. We define

$$\text{im } A = \text{im } T_A = \{\mathbf{b} \in \mathbb{R}^m : \mathbf{b} = A\mathbf{x} \quad \text{for some} \quad \mathbf{x} \in \mathbb{R}^n\} = \text{col}(A).$$

$\text{im}\, T$ is a subspace of W.

Images of linear maps

Let $T : V \to W$ be a linear map.

Proposition-Definition

The **image** of T is the set of all function values of T, that is,

$$\text{im}(T) = \{\mathbf{w} \in W : \mathbf{w} = T(\mathbf{v}) \quad \text{for some} \quad \mathbf{v} \in V\} \subseteq W.$$

Let $A \in M_{mn}(\mathbb{R})$ & $T_A : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be the assoc linear map. We define

$$\text{im } A = \text{im } T_A = \{\mathbf{b} \in \mathbb{R}^m : \mathbf{b} = A\mathbf{x} \quad \text{for some} \quad \mathbf{x} \in \mathbb{R}^n\} = \text{col}(A).$$

$\text{im}\, T$ is a subspace of W.

Remark Proof omitted, but note we already know $\text{col}(A)$ is a subspace as it is the span of the columns of A.

Verifying whether or not vectors lie in the image

Verifying whether or not vectors lie in the image

Example

Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 2 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 3 \\ 1 \\ 3 \end{pmatrix}.$

Verifying whether or not vectors lie in the image

Example

Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 2 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 3 \\ 1 \\ 3 \end{pmatrix}$. Is $\mathbf{b} \in \operatorname{im} A$?

Verifying whether or not vectors lie in the image

Example

Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 2 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 3 \\ 1 \\ 3 \end{pmatrix}$. Is $\mathbf{b} \in \text{im } A$?

Solution

The question amounts to asking: Can we write $\mathbf{b} = A\mathbf{x}$ *for some* $\mathbf{x} \in \mathbb{R}^2$*?*
i.e.

Verifying whether or not vectors lie in the image

Example

Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 2 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 3 \\ 1 \\ 3 \end{pmatrix}$. Is $\mathbf{b} \in \operatorname{im} A$?

Solution

The question amounts to asking: Can we write $\mathbf{b} = A\mathbf{x}$ *for some* $\mathbf{x} \in \mathbb{R}^2$*?*
i.e. Can we solve $A\mathbf{x} = \mathbf{b}$*.*

Finding Bases for Kernels & Images.

Example

Let $\begin{pmatrix}1 & 2 & 3 & 1\\ 2 & 4 & 7 & 1\\ 1 & 2 & 2 & 2\end{pmatrix}$. Find bases for $\ker(A)$ and $\operatorname{im}(A) = \operatorname{col}(A)$.

Solution

$$\begin{pmatrix}1 & 2 & 3 & 1\\ 2 & 4 & 7 & 1\\ 1 & 2 & 2 & 2\end{pmatrix} \longrightarrow \begin{pmatrix}1 & 2 & 3 & 1\\ 0 & 0 & 1 & -1\\ 0 & 0 & -1 & 1\end{pmatrix} \longrightarrow \begin{pmatrix}1 & 2 & 3 & 1\\ 0 & 0 & 1 & -1\\ 0 & 0 & 0 & 0\end{pmatrix} = U$$

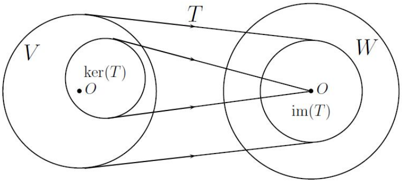
The row echelon form U has first & third columns leading.

Solution (Continued)

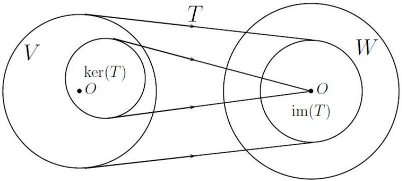
Solution (Continued)

Why study kernels & image?

Why study kernels & image?

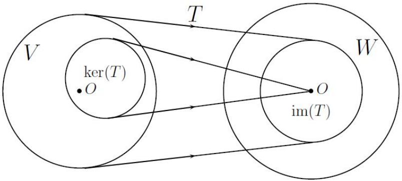


Why study kernels & image?



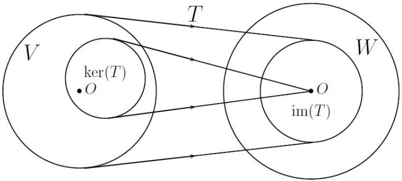
im T tells you about existence of solutions.

Why study kernels & image?



$\operatorname{im} T$ tells you about existence of solutions. $T\mathbf{x} = \mathbf{b}$ has a solution iff $\mathbf{b} \in \operatorname{im} T$.

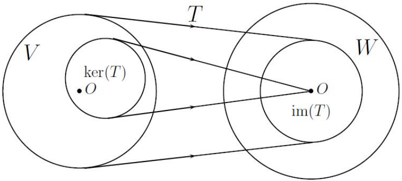
Why study kernels & image?



im T tells you about existence of solutions. $T\mathbf{x} = \mathbf{b}$ has a solution iff $\mathbf{b} \in \operatorname{im} T$.

ker T tells you about uniqueness of solutions.

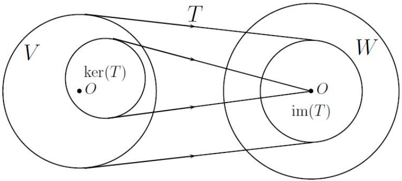
Why study kernels & image?



im T tells you about existence of solutions. $T\mathbf{x} = \mathbf{b}$ has a solution iff $\mathbf{b} \in \text{im}\, T$.

ker T tells you about uniqueness of solutions. $T\mathbf{x} = \mathbf{0}$ has unique solution $\mathbf{x} = \mathbf{0}$ iff $\ker T = \mathbf{0}$.

Why study kernels & image?



im T tells you about existence of solutions. $T\mathbf{x} = \mathbf{b}$ has a solution iff $\mathbf{b} \in \operatorname{im} T$.

ker T tells you about uniqueness of solutions. $T\mathbf{x} = \mathbf{0}$ has unique solution $\mathbf{x} = \mathbf{0}$ iff $\ker T = \mathbf{0}$. We'll see later, that it also tells you about solutions to $T\mathbf{x} = \mathbf{b}$.

Rank and Nullity

Rank and Nullity

Let $T : V \longrightarrow W$ be a linear map & A a matrix with associated linear map T_A.

Definition

- The **nullity** of T is $\text{nullity}(T) = \dim \ker(T)$.

Rank and Nullity

Let $T : V \longrightarrow W$ be a linear map & A a matrix with associated linear map T_A.

Definition

- The **nullity** of T is $\text{nullity}(T) = \dim \ker(T)$.
- The **nullity** of A is $\text{nullity}(A) = \text{nullity}(T_A) = \dim \ker(A)$.

Rank and Nullity

Let $T : V \longrightarrow W$ be a linear map & A a matrix with associated linear map T_A.

Definition

- The **nullity** of T is $\text{nullity}(T) = \dim \ker(T)$.
- The **nullity** of A is $\text{nullity}(A) = \text{nullity}(T_A) = \dim \ker(A)$.
- The **rank** of T is $\text{rank}(T) = \dim \text{im}(T)$.

Rank and Nullity

Let $T : V \longrightarrow W$ be a linear map & A a matrix with associated linear map T_A.

Definition

- The **nullity** of T is $\text{nullity}(T) = \dim \ker(T)$.
- The **nullity** of A is $\text{nullity}(A) = \text{nullity}(T_A) = \dim \ker(A)$.
- The **rank** of T is $\text{rank}(T) = \dim \text{im}(T)$.
- The **rank** of A is $\text{rank}(A) = \text{rank}(T_A) = \dim \text{im}(A)$.

Rank, Nullity example

Rank, Nullity example

Example (Continued from the example on p.30)

Let $A = \begin{pmatrix} 1 & 2 & 3 & 1 \\ 2 & 4 & 7 & 1 \\ 1 & 2 & 2 & 2 \end{pmatrix}$.

Rank, Nullity example

Example (Continued from the example on p.30)

Let $A = \begin{pmatrix} 1 & 2 & 3 & 1 \\ 2 & 4 & 7 & 1 \\ 1 & 2 & 2 & 2 \end{pmatrix}$. Find nullity$(A)$ and rank(A).

Rank, Nullity example

Example (Continued from the example on p.30)

Let $A = \begin{pmatrix} 1 & 2 & 3 & 1 \\ 2 & 4 & 7 & 1 \\ 1 & 2 & 2 & 2 \end{pmatrix}$. Find nullity$(A)$ and rank(A).

Solution

Rank, Nullity example

Example (Continued from the example on p.30)

Let $A = \begin{pmatrix} 1 & 2 & 3 & 1 \\ 2 & 4 & 7 & 1 \\ 1 & 2 & 2 & 2 \end{pmatrix}$. Find nullity$(A)$ and rank(A).

Solution

A basis for im A was

Rank, Nullity example

Example (Continued from the example on p.30)

Let $A = \begin{pmatrix} 1 & 2 & 3 & 1 \\ 2 & 4 & 7 & 1 \\ 1 & 2 & 2 & 2 \end{pmatrix}$. Find nullity($A$) and rank($A$).

Solution

A basis for im A was

Recall a basis for ker A had

Rank, Nullity example

Example (Continued from the example on p.30)

Let $A = \begin{pmatrix} 1 & 2 & 3 & 1 \\ 2 & 4 & 7 & 1 \\ 1 & 2 & 2 & 2 \end{pmatrix}$. Find nullity($A$) and rank($A$).

Solution

A basis for im A was

Recall a basis for ker A had

Note that the basis vectors for im A corresponded to the leading columns of the row-echelon form U whilst the basis vectors for ker A corresponded to the non-leading columns.

Rank & Nullity from the row-echelon form

Rank & Nullity from the row-echelon form

The previous examples illustrates

Rank & Nullity from the row-echelon form

The previous examples illustrates

Key Lemma

Let A be an $m \times n$ matrix which reduces to a row-echelon form U.

Rank & Nullity from the row-echelon form

The previous examples illustrates

Key Lemma

Let A be an $m \times n$ matrix which reduces to a row-echelon form U.

1. nullity(A) = no. parameters in the general soln to $A\mathbf{x} = \mathbf{0}$

Rank & Nullity from the row-echelon form

The previous examples illustrates

Key Lemma

Let A be an $m \times n$ matrix which reduces to a row-echelon form U.

1. $$\begin{aligned} \text{nullity}(A) &= \text{no. parameters in the general soln to } A\mathbf{x} = \mathbf{0} \\ &= \text{the number of non-leading columns of } U. \end{aligned}$$

Rank & Nullity from the row-echelon form

The previous examples illustrates

Key Lemma

Let A be an $m \times n$ matrix which reduces to a row-echelon form U.

1. $\text{nullity}(A) = \text{no. parameters in the general soln to } A\mathbf{x} = \mathbf{0}$
   $= \text{the number of non-leading columns of } U.$
2. $\text{rank}(A) = \text{the maximal no. independent columns of } A$

Rank & Nullity from the row-echelon form

The previous examples illustrates

Key Lemma

Let A be an $m \times n$ matrix which reduces to a row-echelon form U.

1. $\text{nullity}(A) = \text{no. parameters in the general soln to } A\mathbf{x} = \mathbf{0}$
   $= \text{the number of non-leading columns of } U.$
2. $\text{rank}(A) = \text{the maximal no. independent columns of } A$
   $= \text{the number of leading columns of } U.$

Rank-Nullity Theorem

Rank-Nullity Theorem

Our key lemma gives

Theorem (Rank-nullity Theorem for Matrices)

If A is an $m \times n$ matrix, then

$$\text{rank}(A) + \text{nullity}(A) = n.$$

Rank-Nullity Theorem

Our key lemma gives

Theorem (Rank-nullity Theorem for Matrices)

If A is an $m \times n$ matrix, then

$$rank(A) + nullity(A) = n.$$

Proof.

□

Rank-Nullity Theorem

Our key lemma gives

Theorem (Rank-nullity Theorem for Matrices)

If A is an $m \times n$ matrix, then

$$rank(A) + nullity(A) = n.$$

Proof.

□

This can be used to prove more generally,

Theorem (Rank-nullity Theorem for Linear Maps)

Let $T : V \to W$ be a linear map with V finite dimensional. Then

$$rank(T) + nullity(T) = dim(V).$$

Example of rank-nullity theorem

Example

Let $\mathbf{b} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be the projection map $\text{proj}_{\mathbf{b}}$.

Example

Let $\mathbf{b} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be the projection map $\text{proj}_{\mathbf{b}}$.
Verify the rank-nullity theorem in this case.

Example of rank-nullity theorem

Example

Let $\mathbf{b} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be the projection map $\text{proj}_{\mathbf{b}}$.
Verify the rank-nullity theorem in this case.

Solution

Nature of solutions to $A\mathbf{x} = \mathbf{b}$

Nature of solutions to $A\mathbf{x} = \mathbf{b}$

Our key lemma also gives

Nature of solutions to $A\mathbf{x} = \mathbf{b}$

Our key lemma also gives

Theorem

The equation $A\mathbf{x} = \mathbf{b}$ *has:*

Nature of solutions to $A\mathbf{x} = \mathbf{b}$

Our key lemma also gives

Theorem

The equation $A\mathbf{x} = \mathbf{b}$ *has:*

1. *no solution if* $\mathit{rank}(A) \neq \mathit{rank}([A|\mathbf{b}])$,

Nature of solutions to $A\mathbf{x} = \mathbf{b}$

Our key lemma also gives

Theorem

The equation $A\mathbf{x} = \mathbf{b}$ *has:*

1. *no solution if* $rank(A) \neq rank([A|\mathbf{b}])$, *and*
2. *at least one solution if* $rank(A) = rank([A|\mathbf{b}])$.

Nature of solutions to $A\mathbf{x} = \mathbf{b}$

Our key lemma also gives

Theorem

The equation $A\mathbf{x} = \mathbf{b}$ *has:*

1. *no solution if* $\text{rank}(A) \neq \text{rank}([A|\mathbf{b}])$, *and*
2. *at least one solution if* $\text{rank}(A) = \text{rank}([A|\mathbf{b}])$. *Further,*
   i) *if* $\text{nullity}(A) = 0$ *the solution is unique,*

Nature of solutions to $A\mathbf{x} = \mathbf{b}$

Our key lemma also gives

Theorem

The equation $A\mathbf{x} = \mathbf{b}$ has:

1. *no solution if rank$(A) \neq$ rank$([A|\mathbf{b}])$, and*
2. *at least one solution if rank$(A) =$ rank$([A|\mathbf{b}])$. Further,*
   i) *if nullity$(A) = 0$ the solution is unique, whereas,*
   ii) *if nullity$(A) = \nu > 0$, then the general solution is of the form*

$$\mathbf{x} = \mathbf{x}_p + \lambda_1 \mathbf{k}_1 + \cdots + \lambda_\nu \mathbf{k}_\nu \quad \text{for } \lambda_1, \ldots, \lambda_\nu \in \mathbb{R},$$

where $\mathbf{x}_p$ is any solution of $A\mathbf{x} = \mathbf{b}$, and where $\{\mathbf{k}_1, \ldots, \mathbf{k}_\nu\}$ is a basis for ker(A).

A theoretical application of rank-nullity theorem

Example

Prove that if $T : \mathbb{R}^n \to \mathbb{R}^n$ is linear, then the following are equivalent.

a) For all $\mathbf{b} \in \mathbb{R}^n$, there is at least one solution to $T\mathbf{x} = \mathbf{b}$.

b) For all $\mathbf{b} \in \mathbb{R}^n$, there is at most one solution to $T\mathbf{x} = \mathbf{b}$

A theoretical application of rank-nullity theorem

Example

Prove that if $T : \mathbb{R}^n \to \mathbb{R}^n$ is linear, then the following are equivalent.

a) For all $\mathbf{b} \in \mathbb{R}^n$, there is at least one solution to $T\mathbf{x} = \mathbf{b}$.

b) For all $\mathbf{b} \in \mathbb{R}^n$, there is at most one solution to $T\mathbf{x} = \mathbf{b}$

Solution

Linear Differential Equations

Q In what sense are second order linear differential equations linear?

Linear Differential Equations

Q In what sense are second order linear differential equations linear?
A They involve the linear map $T : \mathcal{C}^2[\mathbb{R}] \to \mathcal{C}[\mathbb{R}]$, where $\mathcal{C}^2[\mathbb{R}]$ is the vector space of all $\mathbb{R}$-valued functions with continuous second derivatives and $\mathcal{C}[\mathbb{R}]$ is the vector space of all continuous $\mathbb{R}$-valued functions—

$$T(y) = a\,\frac{d^2y}{dx^2} + b\,\frac{dy}{dx} + cy, \quad \text{where } a, b, c \in \mathbb{R}.$$

Linear Differential Equations

Q In what sense are second order linear differential equations linear?
A They involve the linear map $T : \mathcal{C}^2[\mathbb{R}] \to \mathcal{C}[\mathbb{R}]$, where $\mathcal{C}^2[\mathbb{R}]$ is the vector space of all $\mathbb{R}$-valued functions with continuous second derivatives and $\mathcal{C}[\mathbb{R}]$ is the vector space of all continuous $\mathbb{R}$-valued functions—

$$T(y) = a\,\frac{d^2y}{dx^2} + b\,\frac{dy}{dx} + cy, \quad \text{where } a, b, c \in \mathbb{R}.$$

In this case, $\ker(T)$ is the solution set of the ODE —

$$a\,\frac{d^2y}{dx^2} + b\,\frac{dy}{dx} + cy = 0, \quad \text{where } a, b, c \in \mathbb{R}.$$

Linear Differential Equations

Q In what sense are second order linear differential equations linear?
A They involve the linear map $T : \mathcal{C}^2[\mathbb{R}] \to \mathcal{C}[\mathbb{R}]$, where $\mathcal{C}^2[\mathbb{R}]$ is the vector space of all $\mathbb{R}$-valued functions with continuous second derivatives and $\mathcal{C}[\mathbb{R}]$ is the vector space of all continuous $\mathbb{R}$-valued functions—

$$T(y) = a\,\frac{d^2y}{dx^2} + b\,\frac{dy}{dx} + cy, \quad \text{where } a, b, c \in \mathbb{R}.$$

In this case, $\ker(T)$ is the solution set of the ODE —

$$a\,\frac{d^2y}{dx^2} + b\,\frac{dy}{dx} + cy = 0, \quad \text{where } a, b, c \in \mathbb{R}.$$

Hence the homogeneous solution is a vector space.

Linear Differential Equations

Q In what sense are second order linear differential equations linear?
A They involve the linear map $T : \mathcal{C}^2[\mathbb{R}] \to \mathcal{C}[\mathbb{R}]$, where $\mathcal{C}^2[\mathbb{R}]$ is the vector space of all $\mathbb{R}$-valued functions with continuous second derivatives and $\mathcal{C}[\mathbb{R}]$ is the vector space of all continuous $\mathbb{R}$-valued functions—

$$T(y) = a\,\frac{d^2y}{dx^2} + b\,\frac{dy}{dx} + cy, \quad \text{where } a, b, c \in \mathbb{R}.$$

In this case, $\ker(T)$ is the solution set of the ODE —

$$a\,\frac{d^2y}{dx^2} + b\,\frac{dy}{dx} + cy = 0, \quad \text{where } a, b, c \in \mathbb{R}.$$

Hence the homogeneous solution is a vector space. Furthermore, it is of dimension 2 i.e. $\text{nullity}(T) = 2$.

Linear Differential Equations

Q In what sense are second order linear differential equations linear?
A They involve the linear map $T : \mathcal{C}^2[\mathbb{R}] \to \mathcal{C}[\mathbb{R}]$, where $\mathcal{C}^2[\mathbb{R}]$ is the vector space of all $\mathbb{R}$-valued functions with continuous second derivatives and $\mathcal{C}[\mathbb{R}]$ is the vector space of all continuous $\mathbb{R}$-valued functions—

$$T(y) = a\,\frac{d^2y}{dx^2} + b\,\frac{dy}{dx} + cy, \quad \text{where } a, b, c \in \mathbb{R}.$$

In this case, $\ker(T)$ is the solution set of the ODE —

$$a\,\frac{d^2y}{dx^2} + b\,\frac{dy}{dx} + cy = 0, \quad \text{where } a, b, c \in \mathbb{R}.$$

Hence the homogeneous solution is a vector space. Furthermore, it is of dimension 2 i.e. $\text{nullity}(T) = 2$. We can also apply similar ideas for the solution to $A\mathbf{x} = \mathbf{b}$ to get the solution to

$$a\,\frac{d^2y}{dx^2} + b\,\frac{dy}{dx} + cy = f(x), \quad \text{where } a, b, c \in \mathbb{R}.$$

Example involving polynomials

Example involving polynomials

To study boundary value problems, it's useful to study linear maps such as the one below.

Example involving polynomials

To study boundary value problems, it's useful to study linear maps such as the one below.

Example

The function $T : \mathbb{P}_2 \longrightarrow \mathbb{R}^2$ is defined by $Tp = \binom{p(1)}{p'(1)}$

a) Prove that T is linear.

b) Find $\ker(T)$.

c) Use the rank-nullity theorem to find $\text{im}(T)$.

Solution

Solution (Continued)